UNCOUNTABLE DIRECT SYSTEMS AND A CHARACTERIZATION OF NON-SEPARABLE PROJECTIVE C^*-ALGEBRAS

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ABSTRACT. We introduce the concept of a direct C_ω^*-system and show that every non-separable unital C^*-algebra is the limit of essentially unique direct C_ω^*-system. This result is then applied to the problem of characterization of projective unital C^*-algebras. It is shown that a non-separable unital C^*-algebra X of density τ is projective if and only if it is the limit of a well ordered direct system $\mathcal{S}_X = \{X_\alpha, i_\alpha^{\alpha+1}, \alpha < \tau\}$ of length τ, consisting of unital projective C^*-subalgebras X_α of X and doubly projective homomorphisms (inclusions) $i_\alpha^{\alpha+1}\colon X_\alpha \to X_{\alpha+1}$, $\alpha < \tau$, so that X_0 is separable and each $i_\alpha^{\alpha+1}$, $\alpha < \tau$, has a separable type. In addition we show that a doubly projective homomorphism $f\colon X \to Y$ of unital projective C^*-algebras has a separable type if and only if there exists a pushout diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\[2pt]
{\scriptstyle p}\big\uparrow & & \big\uparrow{\scriptstyle q} \\[2pt]
X_0 & \xrightarrow{\ f_0\ } & Y_0,
\end{array}
$$

where X_0 and Y_0 are separable unital projective C^*-algebras and the homomorphisms $i_0\colon X_0 \to Y_0$, $p\colon X_0 \to X$ and $q\colon Y_0 \to Y$ are doubly projective. These two results provide a complete characterization of non-separable projective unital C^*-algebras in terms of separable ones.

1. INTRODUCTION

The concept of the direct system in the C^*-algebra theory has been successfully used in a wide range of situations. While playing an important role in different constructions and proofs of various statements, direct systems have been used as a tool of introducing new concepts as well. But perhaps the most significant demonstration of the power of direct systems is a possibility of investigation of complicated C^*-algebras by means of their approximation (via direct systems) by simpler C^*-algebras. Such an approach is standard in almost any category which possesses direct (or dually, inverse) systems. Difficulties in systematic implementation of such a method have variety of sources. If, for

instance, we wish to investigate a particular property of an arbitrarily given C^*-algebra X by analyzing a randomly taken direct system $\mathcal{S}_X$, the limit of which is isomorphic to X, then we immediately face the fundamental problem of choice. Is the information encoded in the direct system $\mathcal{S}_X$ relevant to the property of its limit under consideration? Does there exist a direct system with the same limit which is better designed for detecting that property? Are there effective ways of finding such a system? In other words, if two direct systems $\mathcal{S}_X = \{X_\alpha, i_\alpha^\beta, A\}$ and $\mathcal{S}_Y = \{Y_\alpha, j_\alpha^\beta, A\}$ have isomorphic limits are these systems internally related to each other? Do they, for instance, contain isomorphic subsystems? Trivial examples show that the answer in general is negative. Two direct sequences

$$\mathcal{S}_0 = \left\{ C\left(\{0,1\}^{2n}\right), C\left(\pi_{2n}^{2(n+1)}\right), \omega \right\}$$

and

$$\mathcal{S}_1 = \left\{ C\left(\{0,1\}^{2n+1}\right), C\left(\pi_{2n+1}^{2(n+1)+1}\right), \omega \right\},$$

where $\{0,1\}$ is the two-point discrete space and $\pi_n^k \colon \{0,1\}^k \to \{0,1\}^n$, $k > n$, stands for the natural projection, obviously have the same limit – the C^*-algebra of continuous complex-valued functions of the Cantor discontinuum – but contain no isomorphic subsequences whatsoever.

In Section 3 we introduce (Definition 3.1) the concept of direct C^*_ω-system and prove (Theorem 3.5 and Proposition 3.6) that if a unital C^*-algebra is represented as the limit of two direct C^*_ω-systems, then these systems necessarily contain cofinal isomorphic subsystems. It is important to note that every non-separable unital C^*-algebra is the limit of at least one direct C^*_ω-system (Proposition 3.2). Therefore every non-separable unital C^*-algebra X admits essentially unique direct C^*_ω-system $\mathcal{S}_X = \{X_\alpha, i_\alpha^\beta, A\}$ and we conclude that any information about X is contained in $\mathcal{S}_X$. The remaining problem of recovering such an information is, generally speaking, still quite challenging, but has an explicit technical, and not a philosophical, nature. An effective method of searching for such an information is based on Proposition 2.3.

Actually Theorem 3.5 states much more than it might seem to be the case. Not only it states, as was indicated above, that every two direct C^*_ω-systems with isomorphic limits contain isomorphic cofinal subsystems, but it essentially guarantees that any homomorphism $f \colon \varinjlim \mathcal{S}_X \to \varinjlim \mathcal{S}_Y$ between the limits of two direct C^*_ω-systems $\mathcal{S}_X = \{X_\alpha, i_\alpha^\beta, A\}$ and $\mathcal{S}_Y = \{Y_\alpha, j_\alpha^\beta, A\}$ is itself the limit $f = \varinjlim\{f_\alpha; \alpha \in A_f\}$ of a certain morphism $\{f_\alpha \colon X_\alpha \to Y_\alpha, A_f\} \colon \mathcal{S}_X|A_f \to \mathcal{S}_Y|A_f$, consisting of "level" homomorphisms, between cofinal subsystems of the given ones. Such a phenomenom, as was indicated above, is not possible for direct sequences.

We apply the above outlined results to the problem of characterization of non-separable projective unital C^*-algebras in terms of separable ones. Here is the scheme we follow. First we show (Lemma 5.12) that any non-separable projective unital C^*-algebra X is the limit of a direct C_ω^*-system $\mathcal{S}_X = \{X_\alpha, i_\alpha^\beta, A\}$, where X_α's, $\alpha \in A$, are separable projective unital C^*-subalgebras of X and the unital $*$-homomorphisms $i_\alpha^\beta \colon X_\alpha \to X_\beta$, $\alpha \leq \beta$, $\alpha, \beta \in A$, are inclusions. It should be pointed out here that the converse of this fact fails to be true, i.e. there does exist a non-separable non-projective unital C^*-algebra which is the limit of a direct C_ω^*-system consisting of separable and projective unital C^*-subalgebras. This is how we arrive to the necessity of analyzing inclusion homomorphisms i_α^β. What kind of property of these inclusion homomorphisms must be present in order to guarantee that the limit of a direct system $\mathcal{S}_X = \{X_\alpha, i_\alpha^\beta, A\}$, consisting of projective C^*-subalgebras, is projective? We especially emphasize this step because it is a crucial ingredient of a typical argument based on Theorem 3.5. In our particular situation explanation is simple. The concept of a projective object has an explicit categorical nature and seems logical to anticipate that the required property of inclusion homomorphisms is closely related to it. Consequently it makes sense to examine what does the projectivity of a unital $*$-homomorphism, considered as an object of the category $\mathrm{Mor}(\mathcal{C}_1^*)$ of unital $*$-homomorphisms of unital C^*-algebras, mean. It turns out (Proposition 5.11) that projective objects of the category $\mathrm{Mor}(\mathcal{C}_1^*)$ are precisely doubly projective unital $*$-homomorphisms in the sense of [7].

In Section 5 we establish certain properties of doubly projective homomorphisms and present two characterizations of non-separable projective unital C^*-algebras – one (condition (b) of Theorem 5.13) in terms of direct C_ω^*-systems and the other (condition (c) of Theorem 5.13) in terms of well ordered continuous direct systems. The latter states that a non-separable unital C^*-algebra X of density τ is projective if and only if it is the limit of a well ordered direct system $\mathcal{S}_X = \{X_\alpha, i_\alpha^{\alpha+1}, \alpha < \tau\}$ of length τ, consisting of unital projective C^*-subalgebras X_α of X and doubly projective homomorphisms (inclusions) $i_\alpha^{\alpha+1} \colon X_\alpha \to X_{\alpha+1}$, $\alpha < \tau$, so that X_0 is separable and each $i_\alpha^{\alpha+1}$, $\alpha < \tau$, has a separable type (Definition 5.8).

Obviously this result can not be accepted as the one providing a satisfactory reduction of the non-separable case to the separable one. Of course, everything is fine if the density of X is ω_1 – in such a case all X_α's, $\alpha < \omega_1$, (and not only the very first one, i.e. X_0) are indeed separable. But if the density of X is greater than ω_1, then all X_α's, with $\alpha \geq \omega_1$, are non-separable.

In order to achieve our final goal and complete the reduction, we, in Section 6, analyze doubly projective homomorphisms of separable type between (generally speaking, non-separable) projective unital C^*-algebras. A characterization of such homomorphisms, which is recorded in Theorem 6.8 (see also Corollary 6.9), states that a doubly projective homomorphism $f \colon X \to Y$ of projective

unital C^*-algebras has a separable type if and only if there exists a pushout diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\[2pt]
p\big\uparrow & & \big\uparrow q \\[2pt]
X_0 & \xrightarrow{\ f_0\ } & Y_0,
\end{array}
$$

where X_0 and Y_0 are separable unital projective C^*-algebras and the homomorphisms $i_0\colon X_0 \to Y_0$, $p\colon X_0 \to X$ and $q\colon Y_0 \to Y$ are doubly projective.

Theorems 5.13 and 6.8 together complete the required reduction.

Proofs of above statements are based on some properties of unital free products. These properties are undoubtedly known to the experts in the field. For the readers convenience we discuss them in Section 4.

2. Preliminaries

All C^*-algebras below are assumed to be unital and all $*$-homomorphisms between unital C^*-algebras are also unital. The category formed by such C^*-algebras and homomorphisms is denoted by $\mathcal{C}_1^*$. The density $d(X)$ of a C^*-algebra X is the minimal cardinality of dense subspaces (in a purely topological sense) of X. Thus $d(X) \leq \omega$ (ω denotes the first infinite cardinal number) means that X is separable. The unital C^*-algebra, consisting of only one element, is denoted by $\mathbf{0}$. $\mathbb{C}$ denotes the C^*-algebra of complex numbers.

2.1. **Set-theoretical facts.** For the reader's convenience we begin by presenting necessary set-theoretic facts. Their complete proofs can be found in [4].

Let A be a partially ordered *directed set* (i.e. for every two elements $\alpha, \beta \in A$ there exists an element $\gamma \in A$ such that $\gamma \geq \alpha$ and $\gamma \geq \beta$). We say that a subset $A_1 \subseteq A$ of A *majorates* another subset $A_2 \subseteq A$ of A if for each element $\alpha_2 \in A_2$ there exists an element $\alpha_1 \in A_1$ such that $\alpha_1 \geq \alpha_2$. A subset which majorates A is called *cofinal* in A. A subset of A is said to be a *chain* if every two elements of it are comparable. The symbol $\sup B$, where $B \subseteq A$, denotes the lower upper bound of B (if such an element exists in A). Let now τ be an infinite cardinal number. A subset B of A is said to be τ-*closed* in A if for each chain $C \subseteq B$, with $|C| \leq \tau$, we have $\sup C \in B$, whenever the element $\sup C$ exists in A. Finally, a directed set A is said to be τ-*complete* if for each chain B of elements of A with $|C| \leq \tau$, there exists an element $\sup C$ in A.

The standard example of a τ-complete set can be obtained as follows. For an arbitrary set A let $\exp A$ denote, as usual, the collection of all subsets of A. There is a natural partial order on $\exp A$: $A_1 \geq A_2$ if and only if $A_1 \supseteq A_2$. With this partial order $\exp A$ becomes a directed set. If we consider only those subsets

of the set A which have cardinality $\leq \tau$, then the corresponding subcollection of $\exp A$, denoted by $\exp_\tau A$, serves as a basic example of a τ-complete set.

Proposition 2.1. *Let $\{A_t : t \in T\}$ be a collection of τ-closed and cofinal subsets of a τ-complete set A. If $\mid T \mid \leq \tau$, then the intersection $\cap\{A_t : t \in T\}$ is also cofinal (in particular, non-empty) and τ-closed in A .*

Corollary 2.2. *For each subset B, with $\mid B \mid \leq \tau$, of a τ-complete set A there exists an element $\gamma \in A$ such that $\gamma \geq \beta$ for each $\beta \in B$.*

Proposition 2.3 (Spectral Search)**.** *Let A be a τ-complete set, $L \subseteq A^2$, and suppose the following three conditions are satisfied:*

   Existence: *For each $\alpha \in A$ there exists $\beta \in A$ such that $(\alpha, \beta) \in L$.*
   Majorantness: *If $(\alpha, \beta) \in L$ and $\gamma \geq \beta$, then $(\alpha, \gamma) \in L$.*
   τ-closeness: *Let $\{\alpha_t : t \in T\}$ be a chain in A with $|T| \leq \tau$. If $(\alpha_t, \beta) \in L$ for some $\beta \in A$ and each $t \in T$, then $(\alpha, \beta) \in L$ where $\alpha = \sup\{\alpha_t : t \in T\}$.*

Then the set of all L-reflexive elements of A (an element $\alpha \in A$ is L-reflexive if $(\alpha, \alpha) \in L$) is cofinal and τ-closed in A.

Various applications of the above set-theoretical statements are presented in [4, Chapter 8].

3. Direct systems of unital C^*-algebras

Let us recall definitions of some of the concepts related to the notion of a direct system.

3.1. **Morphisms of direct systems.** A direct system $\mathcal{S} = \{X_\alpha, i_\alpha^\beta, A\}$ of unital C^*-algebras consists of a partially ordered directed indexing set A, unital C^*-algebras X_α, $\alpha \in A$, and unital $*$-homomorphisms $i_\alpha^\beta \colon X_\alpha \to X_\beta$, defined for each pair of indexes $\alpha, \beta \in A$ with $\alpha \leq \beta$, and satisfying the condition $i_\alpha^\gamma = i_\beta^\gamma \circ i_\alpha^\beta$ for each triple of indexes $\alpha, \beta, \gamma \in A$ with $\alpha \leq \beta \leq \gamma$. The limit unital C^*-algebra of the above direct system is denoted by $\varinjlim \mathcal{S}$. For each $\alpha \in A$ there exists a unital $*$-homomorphism $i_\alpha \colon X_\alpha \to \varinjlim \mathcal{S}$ which will be called the α-th limit homomorphism of $\mathcal{S}$.

If A' is a directed subset of the indexing set A, then the subsystem $\{X_\alpha, i_\alpha^\beta, A'\}$ of $\mathcal{S}$ is denoted $\mathcal{S}|A'$.

Suppose that we are given two direct systems (with the same indexing set) $\mathcal{S}_X = \{X_\alpha, i_\alpha^\beta, A\}$ and $\mathcal{S}_Y = \{Y_\alpha, j_\alpha^\beta, A\}$ consisting of unital C^*-algebras and unital $*$-homomorphisms. A *morphism*

$$\{f_\alpha \colon \alpha \in A\} \colon \mathcal{S}_X \to \mathcal{S}_Y$$

of the system $\mathcal{S}_X$ into the system $\mathcal{S}_Y$ is a collection $\{f_\alpha \colon \alpha \in A\}$ of unital $*$-homomorphisms $f_\alpha \colon X_\alpha \to Y_\alpha$, defined for all $\alpha \in A$, such that

$$j_\alpha^\beta \circ f_\alpha = f_\beta \circ i_\alpha^\beta,$$

whenever $\alpha, \beta \in A$ and $\alpha \leq \beta$. In other words, we require (in the above situation) the commutativity of the following diagram

$$
\begin{array}{ccc}
X_\beta & \xrightarrow{\ f_\beta\ } & Y_\beta \\
{\scriptstyle i_\alpha^\beta}\big\uparrow & & \big\uparrow{\scriptstyle j_\alpha^\beta} \\
X_\alpha & \xrightarrow{\ f_\alpha\ } & Y_\alpha
\end{array}
$$

Any morphism $\{f_\alpha \colon \alpha \in A\} \colon \mathcal{S}_X \to \mathcal{S}_Y$ induces the unital $*$-homomorphism, called *the limit homomorphism of the morphism,*

$$\varinjlim \{f_\alpha \colon \alpha \in A\} \colon \varinjlim \mathcal{S}_X \to \varinjlim \mathcal{S}_Y$$

such that $\varinjlim \{f_\alpha \colon \alpha \in A\} \circ i_\alpha = j_\alpha \circ f_\alpha$ for each $\alpha \in A$. This obviously means that all diagrams of the form

$$
\begin{array}{ccc}
\varinjlim \mathcal{S}_X & \xrightarrow{\ \varinjlim\{f_\alpha \colon \alpha \in A\}\ } & \varinjlim \mathcal{S}_X \\
{\scriptstyle i_\alpha}\big\uparrow & & \big\uparrow{\scriptstyle j_\alpha} \\
X_\alpha & \xrightarrow{\ f_\alpha\ } & Y_\alpha
\end{array}
$$

commute.

In particular, if for a direct system $\mathcal{S} = \{X_\alpha, i_\alpha^\beta, A\}$ of unital C^*-algebras and for a unital C^*-algebra Y, we are given unital $*$-homomorphisms $f_\alpha \colon X_\alpha \to Y$ so that $f_\alpha = f_\beta \circ i_\alpha^\beta$ for each $\alpha, \beta \in A$ with $\alpha \leq \beta$, then there exists the unique unital $*$-homomorphism $\varinjlim \{f_\alpha \colon \alpha \in A\} \colon \varinjlim \mathcal{S} \to Y$ such that $f_\alpha = \varinjlim \{f_\alpha \colon \alpha \in A\} \circ i_\alpha$ for each $\alpha \in A$. To see this apply the above observation to the trivial direct system $\mathcal{S}_Y = \{Y_\alpha, j_\alpha^\beta, A\}$, where $Y_\alpha = Y$ and $j_\alpha^\beta = \mathrm{id}_Y$ for each $\alpha, \beta \in A$ with $\alpha \leq \beta$.

In the cases when all homomorphisms $i_\alpha^\beta \colon X_\alpha \to X_\beta$ and $i_\alpha \colon X_\alpha \to \varinjlim \mathcal{S}$ are inclusions we will sometimes identify X_α with its image $i_\alpha(X_\alpha)$ in $\varinjlim \mathcal{S}$ and denote the corresponding direct system shortly by $\mathcal{S} = \{X_\alpha, A\}$.

A direct system $\mathcal{S}_X = \{X_\alpha, i_\alpha^{\alpha+1}, \tau\}$, the indexing set of which is an infinite cardinal number τ, is called well ordered. We say that such a direct system is continuous if for each limit ordinal number $\beta < \tau$ the homomorphism

$$\varinjlim \{i_\alpha^\beta; \alpha < \beta\} \colon \varinjlim \{X_\alpha, i_\alpha^{\alpha+1}, \beta\} \to X_\beta$$

is an isomorphism.

3.2. Direct C^*_τ-systems of C^*-algebras.

The concept of the direct C^*_τ-system, introduced in the following definition, will be used below.

Definition 3.1. Let $\tau \geq \omega$ be a cardinal number. A direct system $\mathcal{S} = \{X_\alpha, i^\beta_\alpha, A\}$ of unital C^*-algebras and unital $*$-homomorphisms is called a *direct C^*_τ-system* if the following conditions are satisfied:

(a) A is a τ-complete set.
(b) Density of X_α is at most τ (i.e. $d(X_\alpha) \leq \tau$), $\alpha \in A$.
(c) The α-th limit homomorphism $i_\alpha \colon X_\alpha \to \varinjlim \mathcal{S}$ is an injective $*$-homomorphism for each $\alpha \in A$.
(d) If $B = \{\alpha_t \colon t \in T\}$ is a chain of elements of A with $|T| \leq \tau$ and $\alpha = \sup B$, then the limit homomorphism $\varinjlim\{i^\alpha_{\alpha_t} \colon t \in T\} \colon \varinjlim (\mathcal{S}_X | B) \to X_\alpha$ is an isomorphism.

Proposition 3.2. *Let τ be an infinite cardinal number. Every unital C^*-algebra X can be represented as the limit of a direct C^*_τ-system $\mathcal{S}_X = \{X_\alpha, i^\beta_\alpha, \exp_\tau d(X)\}$.*

Proof. If $d(X) \leq \tau$, then consider the direct C^*_τ-system $\mathcal{S}_X = \{X_\alpha, i^\beta_\alpha, \exp_\tau d(X)\}$, where $X_\alpha = X$ for each $\alpha \in \exp_\tau d(X)$ and $i^\beta_\alpha = \mathrm{id}_X$ for each $\alpha, \beta \in \exp_\tau d(X)$ with $\alpha \leq \beta$.

If $d(X) > \tau$, then consider any subset Y of X such that $\mathrm{cl}_X Y = X$ and $|Y| = d(X)$. Without loss of generality we may assume that Y contains the unit of X. Each $\alpha \in \exp_\tau d(X)$ can obviously be identified with a subset (denoted by the same letter α) of Y of cardinality $\leq \tau$. Let X_α be the smallest C^*-subalgebra of X containing α. If $\alpha, \beta \in \exp_\tau d(X)$ and $\alpha \leq \beta$, then $\alpha \subseteq \beta$ (as subsets of Y) and consequently $X_\alpha \subseteq X_\beta$. This inclusion map is denoted by $i^\beta_\alpha \colon X_\alpha \to X_\beta$. It is easy to verify that the collection $\mathcal{S}_X = \{X_\alpha, i^\beta_\alpha, \exp_\tau d(X)\}$ is indeed a direct C^*_τ-system such that $\varinjlim \mathcal{S}_X = X$. $\qquad\square$

Lemma 3.3. *If $\mathcal{S}_X = \{X_\alpha, i^\beta_\alpha, A\}$ is a direct C^*_τ-system, then*

$$\varinjlim \mathcal{S}_X = \bigcup \{i_\alpha(X_\alpha) \colon \alpha \in A\}.$$

Proof. Clearly $\bigcup\{i_\alpha(X_\alpha) \colon \alpha \in A\}$ is dense in $\varinjlim \mathcal{S}_X$ (this fact remains true for arbitrary direct systems of C^*-algebras). Consequently, for any point $x \in \varinjlim \mathcal{S}_X$ there exists a sequence $\{x_n \colon n \in \omega\}$, consisting of elements from $\bigcup\{i_\alpha(X_\alpha) \colon \alpha \in A\}$, such that $x = \lim\{x_n \colon n \in \omega\}$. For each $n \in \omega$ choose an index $\alpha_n \in A$ such that $x_n \in i_{\alpha_n}(X_{\alpha_n})$. By Corollary 2.2, there exists an index $\alpha \in A$ such that $\alpha \geq \alpha_n$ for each $n \in \omega$. Since $i_{\alpha_n} = i_\alpha \circ i^\alpha_{\alpha_n}$, it follows that

$$x_n \in i_{\alpha_n}(X_{\alpha_n}) = i_\alpha\left(i^\alpha_{\alpha_n}(X_{\alpha_n})\right) \subseteq i_\alpha(X_\alpha) \quad \text{for each } n \in \omega.$$

Finally, since $i_\alpha(X_\alpha)$ is closed in $\varinjlim \mathcal{S}_X$, it follows that

$$x = \lim\{x_n \colon n \in \omega\} \in i_\alpha(X_\alpha).$$

$\square$

Lemma 3.4. *Let* $\mathcal{S}_X = \{X_\alpha, i_\alpha^\beta, A\}$ *be a direct* C_τ^*-*system and* $f \colon Y \to \varinjlim \mathcal{S}_X$ *be a unital* $*$-*homomorphism of a unital* C^*-*algebra* Y *into the direct limit of* $\mathcal{S}_X$. *If* $d(Y) \leq \tau$, *then there exist an index* $\alpha \in A$ *and a unital* $*$-*homomorphism* $f_\alpha \colon Y \to X_\alpha$ *such that* $f = i_\alpha \circ f_\alpha$.

Proof. Since $d(Y) \leq \tau$, there exists a dense subset $Z = \{z_t \colon t \in T\}$ of Y such that $|T| \leq \tau$. For each $t \in T$ there exists, by Lemma 3.3, an index $\alpha_t \in A$ such that $f(z_t) \in i_{\alpha_t}(X_{\alpha_t})$. Since A is a τ-complete (condition (a) of Definition 3.1), there exists, by Corollary 2.2, an index $\alpha \in A$ such that $\alpha \geq \alpha_t$ for each $t \in T$. As in the proof of Lemma 3.3 we can conclude that

$$f(Z) = f(\{z_t \colon t \in T\}) = \{f(z_t) \colon t \in T\} \subseteq i_\alpha(X_\alpha).$$

Since Z is dense in Y and since $i_\alpha(X_\alpha)$ is closed in $\varinjlim \mathcal{S}_X$ it follows that

$$f(Y) = f(\mathrm{cl}_Y Z) \subseteq \mathrm{cl}_{\varinjlim \mathcal{S}_X} f(Z) \subseteq \mathrm{cl}_{\varinjlim \mathcal{S}_X} i_\alpha(X_\alpha) = i_\alpha(X_\alpha).$$

By condition (c) of Definition 3.1, the α-th limit homomorphism i_α of the direct C_τ^*-system $\mathcal{S}_X$ is an injective unital $*$-homomorphism. Thus the composition $f_\alpha = i_\alpha^{-1} \circ f \colon Y \to X_\alpha$ is a well defined unital $*$-homomorphism. It only remains to note that $i_\alpha \circ f_\alpha = i_\alpha \circ i_\alpha^{-1} \circ f = f$, as required. $\square$

The following statement is one of our main results.

Theorem 3.5. *Let* $\mathcal{S}_X = \{X_\alpha, i_\alpha^\beta, A\}$ *and* $\mathcal{S}_Y = \{Y_\alpha, j_\alpha^\beta, A\}$ *be two direct* C_τ^*-*systems with the same indexing set* A. *If* $f \colon \varinjlim \mathcal{S}_X \to \varinjlim \mathcal{S}_Y$ *is a unital* $*$-*homomorphism between the limit* C^*-*algebras of* $\mathcal{S}_X$ *and* $\mathcal{S}_Y$, *then there exist a cofinal and* τ-*closed subset* $A_f \subseteq A$ *and a morphism*

$$\{f_\alpha \colon X_\alpha \to Y_\alpha, \alpha \in A_f\} \colon \mathcal{S}_X|A_f \to \mathcal{S}_Y|A_f$$

such that $f = \varinjlim\{f_\alpha \colon \alpha \in A_f\}$.

Proof. We perform the spectral search (see Proposition 2.3) with respect to the relation $L_f \subseteq A^2$ which is defined as follows. An ordered pair (α, β) of indeces is an element of L_f if and only if $\alpha \leq \beta$ and there exists a unital $*$-homomorphism $f_\alpha^\beta \colon X_\alpha \to Y_\beta$ such that $f \circ i_\alpha = j_\beta \circ f_\alpha^\beta$, i.e. if the diagram

$$
\begin{array}{ccc}
\varinjlim \mathcal{S}_X & \xrightarrow{\;\;f\;\;} & \varinjlim \mathcal{S}_Y \\[4pt]
{\scriptstyle i_\alpha}\big\uparrow & & \big\uparrow{\scriptstyle j_\beta} \\[4pt]
X_\alpha & \xrightarrow{\;\;f_\alpha^\beta\;\;} & Y_\beta
\end{array}
$$

commutes. Let us verify conditions of Proposition 2.3.

Existence. For each $\alpha \in A$ we need to find an index $\beta \in A$ such that $(\alpha, \beta) \in L_f$. Indeed, according to condition (b) of Definition 3.1, $d(X_\alpha) \leq \tau$. Consider the unital $*$-homomorphism $f \circ i_\alpha \colon X_\alpha \to \varinjlim \mathcal{S}_Y$. By Lemma 3.4, there exist an index $\beta \in A$ (which, without loss of generality, may be assumed to be greater than α) and a unital $*$-homomorphism $f_\alpha^\beta \colon X_\alpha \to Y_\beta$ such that $f \circ i_\alpha = j_\beta \circ f_\alpha^\beta$. This obviously means that $(\alpha, \beta) \in L_f$.

Majorantness. Let $(\alpha, \beta) \in L_f$ and $\gamma \geq \beta$. In order to show that $(\alpha, \gamma) \in L_f$, consider the composition $f_\alpha^\gamma = j_\beta^\gamma \circ f_\alpha^\beta \colon X_\alpha \to Y_\gamma$, where the unital $*$-homomorphism $f_\alpha^\beta \colon X_\alpha \to Y_\beta$ is supplied by the condition $(\alpha, \beta) \in L_f$. Clearly $j_\gamma \circ f_\alpha^\gamma = j_\gamma \circ j_\beta^\gamma \circ f_\alpha^\beta = j_\alpha \circ f_\alpha^\beta = f \circ i_\alpha$. This shows that $(\alpha, \gamma) \in L_f$.

τ-closeness. Let $B = \{\alpha_t \colon t \in T\}$ be a chain of elements in A with $|T| \leq \tau$. Suppose that $(\alpha_t, \beta) \in L_f$ for some $\beta \in A$ and each $t \in T$. We need to show that $(\alpha, \beta) \in L_f$, where $\alpha = \sup\{\alpha_t \colon t \in T\}$. First observe that if $\alpha_t \leq \alpha_s$ for $t, s \in T$, then

$$j_\beta \circ f_{\alpha_t}^\beta = f \circ i_{\alpha_t} = f \circ i_{\alpha_s} \circ i_{\alpha_t}^{\alpha_s} = j_\beta \circ f_{\alpha_s}^\beta \circ i_{\alpha_t}^{\alpha_s}.$$

Since, by condition (c) of Definition 3.1, the β-th limit $*$-homomorphism j_β of the direct system $\mathcal{S}_Y$ is injective, it follows that $f_{\alpha_t}^\beta = f_{\alpha_s}^\beta \circ i_{\alpha_t}^{\alpha_s}$. This means that the collection $\{f_{\alpha_t}^\beta \colon t \in T\}$ forms a morphism of the subsystem $\mathcal{S}_X|B$ of the direct C_τ^*-system $\mathcal{S}_X$ into the C^*-algebra Y_β. Consider (see Subsection 3.1) the unital $*$-homomorphism

$$\varinjlim\{f_{\alpha_t}^\beta \colon t \in T\} \colon \varinjlim(\mathcal{S}_X|B) \to Y_\beta.$$

Finally, applying condition (d) of Definition 3.1, we define the unital $*$-homomorphism $f_\alpha^\beta \colon X_\alpha \to Y_\beta$ as the composition

$$X_\alpha \xrightarrow{\left(\varinjlim\{i_{\alpha_t}^\alpha \colon t \in T\}\right)^{-1}} \varinjlim(\mathcal{S}_X|B) \xrightarrow{\varinjlim\{f_{\alpha_t}^\beta \colon t \in T\}} Y_\beta.$$

The straightforward verification shows that f_α^β indeed satisfies the required equality $f \circ i_\alpha = j_\beta \circ f_\alpha^\beta$ and, consequently, $(\alpha, \beta) \in L_f$.

Now, by applying Proposition 2.3, we conclude that the set A_f of L_f-reflexive elements is cofinal and τ-closed in A. Observe that an element $\alpha \in A$ is L_f-reflexive if and only if there exists a unital $*$-homomorphism $f_\alpha \colon X_\alpha \to Y_\alpha$ such that $f \circ i_\alpha = j_\alpha \circ f_\alpha$.

It follows from the above construction that the collection

$$\{f_\alpha \colon X_\alpha \to Y_\alpha, \alpha \in A_f\} \colon \mathcal{S}_X|A_f \to \mathcal{S}_X|A_f$$

is indeed a morphism between the systems $\mathcal{S}_X|A_f$ and $\mathcal{S}_Y|A_f$ such that $f = \varinjlim\{f_\alpha \colon \alpha \in A_f\}$. $\qquad\square$

Proposition 3.6. *If $f \colon \varinjlim \mathcal{S}_X \to \varinjlim \mathcal{S}_Y$ is a unital$*$-isomorphism between the limit C^*-algebras of direct C_τ^*-systems $\mathcal{S}_X = \{X_\alpha, i_\alpha^\beta, A\}$ and $\mathcal{S}_Y = \{Y_\alpha, j_\alpha^\beta, A\}$*

with the same indexing set, then there exist a cofinal and τ-closed subset $A_f \subseteq A$ and a morphism

$$\{f_\alpha \colon X_\alpha \to Y_\alpha, \alpha \in A_f\} \colon \mathcal{S}_X|A_f \to \mathcal{S}_Y|A_f$$

such that $f = \varinjlim\{f_\alpha \colon \alpha \in A_f\}$ and f_α is a unital $*$-isomorphism for each $\alpha \in A_f$.

Proof. By Theorem 3.5, applied to the unital $*$-homomorphism $f \colon \varinjlim \mathcal{S}_X \to \varinjlim \mathcal{S}_Y$, there exist a cofinal and τ-closed subset $\widetilde{A}_f \subseteq A$ and a morphism

$$\{f_\alpha \colon X_\alpha \to Y_\alpha, \alpha \in \widetilde{A}_f\} \colon \mathcal{S}_X|\widetilde{A}_f \to \mathcal{S}_Y|\widetilde{A}_f$$

such that $f = \varinjlim\{f_\alpha \colon \alpha \in \widetilde{A}_f\}$.

Similarly, by Theorem 3.5, applied to the unital $*$-homomorphism $f^{-1} \colon \varinjlim \mathcal{S}_Y \to \varinjlim \mathcal{S}_X$ (recall that f is a unital $*$-isomorphism), there exist a cofinal and τ-closed subset $\widetilde{A}_{f^{-1}} \subseteq A$ and a morphism

$$\{g_\alpha \colon Y_\alpha \to X_\alpha, \alpha \in \widetilde{A}_{f^{-1}}\} \colon \mathcal{S}_Y|\widetilde{A}_{f^{-1}} \to \mathcal{S}_X|\widetilde{A}_{f^{-1}}$$

such that $f^{-1} = \varinjlim\{g_\alpha \colon \alpha \in \widetilde{A}_{f^{-1}}\}$.

By Proposition 2.1, the intersection $A_f = \widetilde{A}_f \cap \widetilde{A}_{f^{-1}}$ is still cofinal and τ-closed subset of A. Note that for each $\alpha \in A_f$ we have two unital $*$-homomorphisms $f_\alpha \colon X_\alpha \to Y_\alpha$ and $g_\alpha \colon Y_\alpha \to X_\alpha$ satisfying the equalities $f \circ i_\alpha = j_\alpha \circ f_\alpha$ and $f^{-1} \circ j_\alpha = i_\alpha \circ g_\alpha$. Consequently, having also in mind condition (c) of Definition 3.1, we have

$$g_\alpha \circ f_\alpha = g_\alpha \circ j_\alpha^{-1} \circ j_\alpha \circ f_\alpha = g_\alpha \circ j_\alpha^{-1} \circ f \circ i_\alpha = i_\alpha^{-1} \circ i_\alpha \circ g_\alpha = j_\alpha^{-1} \circ f \circ i_\alpha =$$

$$i_\alpha^{-1} \circ f^{-1} \circ j_\alpha \circ j_\alpha^{-1} \circ f \circ i_\alpha = i_\alpha^{-1} \circ f^{-1} \circ f \circ i_\alpha = i_\alpha^{-1} \circ i_\alpha = \mathrm{id}_{X_\alpha}.$$

Similarly, $f_\alpha \circ g_\alpha = \mathrm{id}_{Y_\alpha}$. This obviously means that both f_α and g_α are $*$-isomorphisms (inverses of each other). $\qquad\square$

4. Unital free products of unital C^*-algebras and their direct C^*_ω-systems

Definition and various properties of (amalgamated) free products of C^*-algebras can be found in [3], [5], [6], [1], [2], [11]. Below we consider only the unital free products of unital C^*-algebras (see [2, Example 1.3(f)]).

Recall that the unital free product of a collection $\{X_t \colon t \in T\}$ of unital C^*-algebras (i.e. the amalgamated free product over the common unit of X_t's) is the unital C^*-algebra $\bigstar_\mathbb{C}\{X_t \colon t \in T\}$, together with unital injective $*$-homomorphisms $\pi_t \colon X_t \to \bigstar_\mathbb{C}\{X_t \colon t \in T\}$, $t \in T$, satisfying the following universality property:

$\left(\bigstar_{\mathbb{C}} \right)$ for any unital $*$-homomorphisms $f_t \colon X_t \to Y$, $t \in T$, into any unital C^*-algebra Y, there exists a unique unital $*$-homomorphism
$$\bigstar_{\mathbb{C}}\{f_t \colon t \in T\} \colon \bigstar_{\mathbb{C}}\{X_t \colon t \in T\} \to Y \text{ such that } f \circ \pi_t = f_t, \, t \in T.$$

For the case $|T| = 2$ this universality property of the unital free products is explicitly stated by several authors (see, for instance, [5, p. 81] [6, p. 156], [7, p. 89], [10, 2.2. Pushouts]). The existence of unital free products of infinite (uncountable) collections of unital C^*-algebras is proved in [1, Theorem 3.1] (see also [2, Example 1.3(f)]). Even though the property $\left(\bigstar_{\mathbb{C}} \right)$ is not explicitly stated in [1], it can be extracted from the proofs provided there.

An alternative approach for establishing the property $\left(\bigstar_{\mathbb{C}} \right)$ for arbitrary T is standard, although less constructive. It is based on the following observation. The unital direct product $(X_1 \bigstar_{\mathbb{C}} X_2, \pi_1, \pi_2)$ of two unital C^*-algebras X_1 and X_2 is precisely the coproduct of the objects X_1 and X_2 in the category $\mathcal{C}_1$ (see [8, p. 63]). Also note that the category $\mathcal{C}_1$ is the category with the initial object – namely, the C^*-algebra $\mathbb{C}$. These two facts suffice [8, Section III.5] to conclude the existence of unital free products of finite collections of unital C^*-algebras.

For an arbitrary indexing set T, consider the directed set $\exp_{<\omega} T$ of all finite subsets of T with the natural partial order generated by the inclusion relation. Next consider the direct system

$$\mathcal{S} = \left\{ \bigstar_{\mathbb{C}}\{X_t \colon t \in A\}, \pi_A^B, A, B \in \exp_{<\omega} T \right\},$$

consisting of the unital free products $\bigstar_{\mathbb{C}}\{X_t \colon t \in A\}$ of finite subcollections and associated injective unital $*$-homomorphisms

$$\pi_A^B \colon \bigstar_{\mathbb{C}}\{X_t \colon t \in A\} \to \bigstar_{\mathbb{C}}\{X_t \colon t \in B\}, A \subseteq B, A, B \in \exp_{<\omega} T.$$

The limit $\varinjlim \mathcal{S}$ of this direct system is in fact the unital free product of the given collection. Unital injective $*$-homomorphisms π_t, participating in the definition of unital free products, are precisely the t-th limit $*$-homomorphisms of the direct system $\mathcal{S}$ (see Subsection 3.1).

To see that the property $\left(\bigstar_{\mathbb{C}} \right)$ is satisfied for so defined unital free products, consider unital $*$-homomorphisms

$$\bigstar_{\mathbb{C}}\{f_t \colon t \in A\} \colon \bigstar_{\mathbb{C}}\{X_t \colon t \in A\} \to Y,$$

uniquely defined (in the above discussed case of the unital free products of finite collections of unital C^*-algebras) for each finite subset A of T. It follows that

$$\bigstar_{\mathbb{C}}\{f_t \colon t \in A\} = \bigstar_{\mathbb{C}}\{f_t \colon t \in B\} \circ \pi_A^B$$

whenever $A, B \in \exp_{<\omega} T$ and $A \subseteq B$. This guarantees (see Subsection 3.1) that the limit $*$-homomorphism

$$\bigstar_{\mathbb{C}}\{f_t \colon t \in T\} = \varinjlim\{\bigstar_{\mathbb{C}}\{f_t \colon t \in A\} \colon A \in \exp_{<\omega} T\} \colon \varinjlim \mathcal{S} \to Y$$

satisfies the required equalities

$$\bigstar_{\mathbb{C}}\{f_t \colon t \in T\} \circ \pi_t = f_t, t \in T.$$

We now state some of the properties of the unital free products which will be needed in later sections.

Lemma 4.1. *If $S \subseteq T$, then $\bigstar_{\mathbb{C}}\{X_t \colon t \in T\}$ is canonically isomorphic to $(\bigstar_{\mathbb{C}}\{X_t \colon t \in S\}) \bigstar_{\mathbb{C}} (\bigstar_{\mathbb{C}}\{X_t \colon t \in T - S\})$.*

Proof. Let

$$\pi_1 \colon \bigstar_{\mathbb{C}}\{X_t \colon t \in S\} \to (\bigstar_{\mathbb{C}}\{X_t \colon t \in S\}) \bigstar_{\mathbb{C}} (\bigstar_{\mathbb{C}}\{X_t \colon t \in T - S\})$$

and

$$\pi_2 \colon \bigstar_{\mathbb{C}}\{X_t \colon t \in T - S\} \to (\bigstar_{\mathbb{C}}\{X_t \colon t \in S\}) \bigstar_{\mathbb{C}} (\bigstar_{\mathbb{C}}\{X_t \colon t \in T - S\})$$

denote the canonical inclusions (see [1, Theorem 3.1]).

Let

$$\pi_t^S \colon X_t \to \bigstar_{\mathbb{C}}\{X_t \colon t \in S\} \ , \ \pi_t^{T-S} \colon X_t \to \bigstar_{\mathbb{C}}\{X_t \colon t \in T - S\}$$

and

$$\pi_t^T \colon X_t \to \bigstar_{\mathbb{C}}\{X_t \colon t \in T\}$$

also denote canonical inclusions into the corresponding unital free products.

Now consider the homomorphisms

$$\bigstar_{\mathbb{C}}\{\pi_t^T \colon t \in S\} \colon \bigstar_{\mathbb{C}}\{X_t \colon t \in S\} \to \bigstar_{\mathbb{C}}\{X_t \colon t \in T\}$$

and

$$\bigstar_{\mathbb{C}}\{\pi_t^T \colon t \in T - S\} \colon \bigstar_{\mathbb{C}}\{X_t \colon t \in T - S\} \to \bigstar_{\mathbb{C}}\{X_t \colon t \in T\}.$$

These two homomorphisms define the unique unital $*$-homomorphism

$$f \colon (\bigstar_{\mathbb{C}}\{X_t \colon t \in S\}) \bigstar_{\mathbb{C}} (\bigstar_{\mathbb{C}}\{X_t \colon t \in T - S\}) \to \bigstar_{\mathbb{C}}\{X_t \colon t \in T\}$$

such that

(4.1) $$f \circ \pi_1 = \bigstar_{\mathbb{C}}\{\pi_t^T \colon t \in S\}$$

and

(4.2) $$f \circ \pi_2 = \bigstar_{\mathbb{C}}\{\pi_t^T \colon t \in T - S\}.$$

Here $f = (\bigstar_{\mathbb{C}}\{\pi_t^T \colon t \in S\}) \bigstar_{\mathbb{C}} (\bigstar_{\mathbb{C}}\{\pi_t^T \colon t \in T - S\})$.

Similarly consider the unique unital $*$-homomorphism

$$g\colon \bigstar_{\mathbb{C}}\{X_t\colon t \in T\} \to (\bigstar_{\mathbb{C}}\{X_t\colon t \in S\}) \bigstar_{\mathbb{C}} (\bigstar_{\mathbb{C}}\{X_t\colon t \in T - S\})$$

satisfying the equalities

(4.3)
$$g \circ \pi_t^T = \begin{cases} \pi_1 \circ \pi_t^S \ , & \text{if } t \in S, \\ \pi_2 \circ \pi_t^{T-S}, & \text{if } t \in T - S. \end{cases}$$

Next observe that if $t \in S$, then

(4.4)
$$f \circ g \circ \pi_t^T \overset{(4.3)}{=} f \circ \pi_1 \circ \pi_t^S \overset{(4.1)}{=} \bigstar_{\mathbb{C}}\{\pi_t^T\colon t \in S\} \circ \pi_t^S = \pi_t^T.$$

Similarly, if $t \in T - S$, then

(4.5)
$$f \circ g \circ \pi_t^T \overset{(4.3)}{=} f \circ \pi_2 \circ \pi_t^{T-S} \overset{(4.2)}{=} \bigstar_{\mathbb{C}}\{\pi_t^T\colon t \in T - S\} \circ \pi_t^{T-S} = \pi_t^T.$$

Now observe that (4.4) and (4.5) guarantee the validity of the equality

(4.6)
$$f \circ g = \mathrm{id}_{\bigstar_{\mathbb{C}}\{X_t\colon t \in T\}}.$$

In order to prove the equality

(4.7)
$$g \circ f = \mathrm{id}_{(\bigstar_{\mathbb{C}}\{\pi_t^T\colon t \in S\}) \bigstar_{\mathbb{C}} (\bigstar_{\mathbb{C}}\{\pi_t^T\colon t \in T - S\})}$$

it suffices to show that

(4.8)
$$g \circ f \circ \pi_1 = \pi_1$$

and

(4.9)
$$g \circ f \circ \pi_2 = \pi_2.$$

Note that (4.8) follows from the following observation ($t \in S$):

(4.10)
$$g \circ f \circ \pi_1 \circ \pi_t^S \overset{(4.1)}{=} g \circ \bigstar_{\mathbb{C}}\{\pi_t^T\colon t \in S\} \circ \pi_t^S = g \circ \pi_t^T \overset{(4.3)}{=} \pi_1 \circ \pi_t^S.$$

Similarly (4.9) follows from the following observation ($t \in T - S$):

(4.11)
$$g \circ f \circ \pi_2 \circ \pi_t^{T-S} \overset{(4.2)}{=} g \circ \bigstar_{\mathbb{C}}\{\pi_t^T\colon t \in S\} \circ \pi_t^{T-S} = g \circ \pi_t^{T-S} \overset{(4.3)}{=} \pi_2 \circ \pi_t^{T-S}.$$

This finishes proof of (4.7).

It only remains to note that, by (4.6) and (4.7), both f and g are isomorphisms as required. $\square$

Lemma 4.2. *If $S \subseteq T$, then the unital $*$-homomorphism*

$$\pi_S^T = \bigstar_{\mathbb{C}}\{\mathrm{id}_{X_t}\colon t \in S\}\colon \bigstar_{\mathbb{C}}\{X_t\colon t \in S\} \to \bigstar_{\mathbb{C}}\{X_t\colon t \in T\}$$

is injective.

Proof. It can be shown, by applying the argument similar to the one used in the proof of Lemma 4.1, that the homomorphism π_S^T coincides with the homomorphism

$$\pi_1\colon \bigstar_{\mathbb{C}}\{X_t\colon t \in S\} \to \left(\bigstar_{\mathbb{C}}\{X_t\colon t \in S\}\right)\bigstar_{\mathbb{C}}\left(\bigstar_{\mathbb{C}}\{X_t\colon t \in T - S\}\right).$$

It only remains to note that π_1 is an inclusion by [1, Theorem 3.1]. $\qquad\square$

Lemma 4.3. *If $\{T_\alpha\colon \alpha < \tau\}$ is an increasing well ordered collection of subsets of T and $T = \cup\{T_\alpha\colon \alpha < \tau\}$, then $\bigstar_{\mathbb{C}}\{X_t\colon t \in T\}$ is canonically isomorphic to the direct limit of the well ordered direct system $\left\{\bigstar_{\mathbb{C}}\{X_t\colon t \in T_\alpha\}, \pi_{T_\alpha}^{T_{\alpha+1}}, \tau\right\}$*

Proof. For each $\alpha < \tau$ consider the unital $*$-homomorphism

$$\pi_{T_\alpha}^T\colon \bigstar_{\mathbb{C}}\{X_t\colon t \in T_\alpha\} \to \bigstar_{\mathbb{C}}\{X_t\colon t \in T\},$$

defined in Lemma 4.2. Clearly $\pi_{T_{\alpha+1}}^T \circ \pi_{T_\alpha}^{T_{\alpha+1}} = \pi_{T_\alpha}^T$ for each $\alpha < \tau$. Consider the unique unital $*$-homomorphism (see Subsection 3.1)

$$f\colon \varinjlim\left\{\bigstar_{\mathbb{C}}\{X_t\colon t \in T_\alpha\}, \pi_{T_\alpha}^{T_{\alpha+1}}, \tau\right\} \to \bigstar_{\mathbb{C}}\{X_t\colon t \in T\}$$

such that $f \circ \pi_\alpha = \pi_{T_\alpha}$ for each $\alpha < \tau$ (here

$$\pi_\alpha\colon \bigstar_{\mathbb{C}}\{X_t\colon t \in T_\alpha\} \to \varinjlim\left\{\bigstar_{\mathbb{C}}\{X_t\colon t \in T_\alpha\}, \pi_{ST_\alpha}^{T_{\alpha+1}}, \tau\right\}$$

denotes the α-th limit injection of the above direct system). Applying property $(\bigstar_{\mathbb{C}})$ it is easy to see that f is an isomorphism. $\qquad\square$

Finally we record the following statement.

Proposition 4.4. *Let $\{X_t\colon t \in T\}$ be an infinite collection of unital C^*-algebras. Then the collection $\left\{\bigstar_{\mathbb{C}}\{X_t\colon t \in S\}, \pi_S^R, S, R \in \exp_{<\omega} T\right\}$, consisting of the unital free products of finite subcollections and above defined canonical injections, is a direct system whose direct limit is the unital free product $\bigstar_{\mathbb{C}}\{X_t\colon t \in T\}$.*

If the given collection $\{X_t\colon t \in T\}$ is uncountable and consists of separable unital C^*-algebras, then the collection $\left\{\bigstar_{\mathbb{C}}\{X_t\colon t \in S\}, \pi_S^R, S, R \in \exp_\omega T\right\}$, consisting of the unital free products of countable subcollections and above defined canonical injections, is a direct C_ω^*-system of the unital free product $\bigstar_{\mathbb{C}}\{X_t\colon t \in T\}$.

Proof. The first part of this statement follows from the above given definition of unital free products. In order to prove the second part we need to show that $\left\{ \bigstar_{\mathbb{C}}\{X_t \colon t \in S\}, \pi_S^R, S, R \in \exp_\omega T \right\}$ is a direct C_ω^*-system associated with the unital free product $\bigstar_{\mathbb{C}}\{X_t \colon t \in T\}$. Let us verify condition (a)–(d) of Definition 3.1. Condition (a) is obvious since the set $\exp_\omega T$ is ω-complete. Conditions (c) and (d) follow from Lemmas 4.2 and 4.3. Finally condition (b), i.e. the fact that $\bigstar_{\mathbb{C}}\{X_t \colon t \in S\}$ is separable for a countable subset $S \subseteq T$, follows from [1, Theorem 3.1]. $\qquad\square$

Remark 4.5. The fact that the homomorphism π_S^T, indicated in Lemma 4.2, is injective can be significantly strengthened in the situation when each X_t admits a unital $*$-homomorphism $\varphi_t \colon X_t \to \mathbb{C}$. Indeed, in such a case, we can choose an index $s_0 \in S$ and view the homomorphism φ_t as a unital $*$-homomorphism of X_t into X_{s_0}. Next consider the unital $*$-homomorphism

$$g_S^T = \bigstar_{\mathbb{C}}\{g_t \colon t \in T\} \colon \bigstar_{\mathbb{C}}\{X_t \colon t \in T\} \to \bigstar_{\mathbb{C}}\{X_t \colon t \in S\},$$

where

$$g_t = \begin{cases} \mathrm{id}_{X_t} \colon X_t \to X_t \ , & \text{if } t \in S, \\ \varphi_t \colon X_t \to X_{s_0} \ , & \text{if } t \in T - S. \end{cases}$$

It is easy to show that $g_S^T \circ \pi_S^T = \mathrm{id}_{\bigstar_{\mathbb{C}}\{X_t \colon t \in S\}}$. This means that π_S^T is a coretraction and, in particular, is injective.

Lemma 4.6. *Let X be a C^*-algebra admitting a unital $*$-homomorphism into $\mathbb{C}$. If Y is a C^*-subalgebra of X, then Y also admits a unital $*$-homomorphism into $\mathbb{C}$. Projective unital C^*-algebra admits a unital $*$-homomorphism into $\mathbb{C}$.*

Proof. The first part is trivial. If X is a projective unital C^*-algebra, then the projection $\pi_1 \colon X \times \mathbb{C} \to X$ of the direct product $X \times \mathbb{C}$ onto the first coordinate has the inverse, i.e. there exists a unital $*$-homomorphism $i \colon X \to X \times \mathbb{C}$ such that $\pi_1 \circ i = \mathrm{id}_X$. Clearly the projection $\pi_2 \colon X \times \mathbb{C} \to \mathbb{C}$ onto the second coordinate is a unital $*$-homomorphism. It only remains to note that the composition $\pi_2 \circ i \colon X \to \mathbb{C}$ is a unital $*$-homomorphism. $\qquad\square$

5. Basic properties of doubly projective homomorphisms and characterization of projective unital C^*-algebras

Recall that a unital C^*-algebra P is *projective* if for any surjective unital $*$-homomorphism $p \colon X \to Y$ of unital C^*-algebras and for any unital $*$-homomorphism $f \colon P \to Y$ there exists a unital $*$-homomorphism $g \colon P \to X$ such that $p \circ g = f$.

5.1. **Doubly projective homomorphisms.** The concept of doubly projective homomorphism was introduced in [7, Definition 3.1]. In the definition given below we do not assume that X and Y are projective C^*-algebras.

Definition 5.1. A unital $*$-homomorphism $i\colon X \to Y$ of unital C^*-algebras X and Y is *doubly projective* if for any surjective unital $*$-homomorphism $p\colon A \to B$ between unital C^*-algebras A and B and any two unital $*$-homomorphisms $f\colon X \to A$ and $g\colon Y \to B$ with $g \circ i = p \circ f$, there exists a unital $*$-homomorphism $h\colon B \to X$ such that $f = h \circ i$ and $g = p \circ h$. In other words, any commutative square diagram

with surjective p can be completed by the diagonal arrow with commuting triangles.

We need some properties of doubly projective homomorphisms.

Lemma 5.2. *A doubly projective homomorphism $i\colon X \to Y$ of unital C^*-algebras is a coretraction, i.e. there exists a unital $*$-homomorphism $r\colon Y \to X$ such that $r \circ i = \mathrm{id}_X$. In particular, a doubly projective homomorphism is injective.*

Proof. Let $i\colon X \to Y$ be a doubly projective homomorphism. Consider the following commutative diagram

Since i is doubly projective, there exists a unital $*$-homomorphism $r\colon Y \to X$ such that $r \circ i = \mathrm{id}_X$. $\square$

Lemma 5.3. *Let $i\colon X \to Y$ be a doubly projective homomorphism of unital C^*-algebras. Then X is projective if and only if Y is projective.*

Proof. First suppose that X is projective. In order to show that Y is projective, consider a surjective unital $*$-homomorphism $p\colon A \to B$ and a unital $*$-homomorphism $g\colon Y \to B$. Our goal is to find a unital $*$-homomorphism $\tilde{g}\colon Y \to A$ such that $p \circ \tilde{g} = g$. Since X is projective, there exists a unital $*$-homomorphism $f\colon X \to A$ such that $p \circ f = g \circ i$. Since i is doubly projective there exists a unital $*$-homomorphism $\tilde{g}\colon Y \to A$ such that $g = p \circ \tilde{g}$ (and $f = \tilde{g} \circ i$). Obviously $\tilde{g}$ is a required lift of g and, consequently, Y is projective.

Now assume that Y is projective. In order to show that X is projective, consider a surjective unital $*$-homomorphism $p\colon A \to B$ and a unital $*$-homomorphism $f\colon X \to B$. Our goal is to find a unital $*$-homomorphism $\tilde{f}\colon X \to A$ such that $p \circ \tilde{f} = f$. By Lemma 5.2, there exists a unital $*$-homomorphism $r\colon Y \to X$ such that $r \circ i = \mathrm{id}_X$. Consider the composition $g = f \circ r\colon Y \to B$. Since Y is projective, there exists a unital $*$-homomorphism $\tilde{g}\colon Y \to A$ such that $p \circ \tilde{g} = g$. Let $\tilde{f} = \tilde{g} \circ i$. It only remains to note that $p \circ \tilde{f} = p \circ \tilde{g} \circ i = g \circ i = f \circ r \circ i = f$. $\qquad\square$

Lemma 5.4. *A finite composition of doubly projective homomorphisms is doubly projective.*

Proof. Let $i_1\colon X_1 \to X_2$ and $i_2\colon X_2 \to X_3$ be doubly projective homomorphisms of unital C^*-algebras. We need to show that the composition $i = i_2 \circ i_1\colon X_1 \to X_3$ is also doubly projective. Consider a surjective unital $*$-homomorphism $p\colon A \to B$ and two unital $*$-homomorphisms $g\colon X_3 \to B$ and $f\colon X_1 \to A$ such that $g \circ i = p \circ f$. Consider the following commutative diagram

$$
\begin{array}{ccc}
B & \xleftarrow{\ g\circ i_2\ } & X_2 \\[2pt]
{\scriptstyle p}\big\uparrow & {}^{f_1}\nearrow\ \ \ \searrow^{f} & \big\uparrow{\scriptstyle i_1} \\[2pt]
A & \xleftarrow{\ f\ } & X_1
\end{array}
$$

Since i_1 is doubly projective and since $(g \circ i_2) \circ i_1 = g \circ i = p \circ f$, there exists a unital $*$-homomorphism $f_1\colon X_2 \to A$ such that $p \circ f_1 = g \circ i_2$ and $f = f_1 \circ i_1$.

Next consider the commutative diagram

$$
\begin{array}{ccc}
B & \xleftarrow{\quad g \quad} & X_3 \\
\Big\uparrow\,{\scriptstyle p} & \diagdown\,{\scriptstyle h} & \Big\downarrow\,{\scriptstyle i_2} \\
A & \xleftarrow{\quad f_1 \quad} & X_2
\end{array}
$$

Since i_2 is doubly projective and since $p \circ f_1 = g \circ i_2$, there exists a unital $*$-homomorphism $h\colon X_3 \to A$ such that $p \circ h = g$ and $f_1 = h \circ i_2$. It only remains to note that

$$
h \circ i = h \circ (i_2 \circ i_1) = (h \circ i_2) \circ i_1 = f_1 \circ i_1 = f.
$$

$\square$

Lemma 5.5. *Let $f\colon X \to Y$ be a doubly projective homomorphism. Suppose that $f = f_2 \circ f_1$, where $f_2\colon Z \to Y$ is a coretraction (i.e. there exists a unital $*$-homomorphism $r\colon Y \to Z$ such that $r \circ f_2 = \mathrm{id}_Z$). Then $f_1\colon X \to Z$ is also doubly projective.*

Proof. Let $p\colon A \to B$ be a surjective homomorphism of unital C^*-algebras. Let also $g\colon X \to A$ and $h\colon Z \to B$ be unital $*$-homomorphisms such that $p \circ g = h \circ f_1$. We need to find a unital $*$-homomorphism $k\colon Z \to A$ such that $k \circ f_1 = g$ and $p \circ k = h$. Note that

$$
p \circ g = h \circ f_1 = h \circ r \circ f_2 \circ f_1 = (h \circ r) \circ f.
$$

Since f is doubly projective, there exists a unital $*$-homomorphism $\widetilde{k}\colon Y \to A$ such that $\widetilde{k} \circ f = g$ and $p \circ \widetilde{k} = h \circ r$. Finally note that the composition $k = \widetilde{k} \circ f_2\colon Z \to A$ has all the required properties. Indeed,

$$
k \circ f_1 = \widetilde{k} \circ f_2 \circ f_1 = \widetilde{k} \circ f = g
$$

and

$$
p \circ k = p \circ \widetilde{k} \circ f_2 = h \circ r \circ f_2 = h.
$$

$\square$

Lemma 5.6. *Let $i\colon X \to Y$ be a unital $*$-homomorphism which is a retract of a unital $*$-homomorphism $i'\colon X' \to Y'$. This means that there exist unital $*$-homomorphisms $\varphi_X\colon X' \to X$, $\varphi_Y\colon Y' \to Y$, $\phi_X\colon X \to X'$ and $\phi_Y\colon Y \to Y'$ such that $i \circ \varphi_X = \varphi_Y \circ i'$, $i' \circ \phi_X = \phi_Y \circ i$, $\varphi_X \circ \phi_X = \mathrm{id}_X$ and $\varphi_Y \circ \phi_Y = \mathrm{id}_Y$. In other words the diagram*

$$Y \xleftarrow{\varphi_Y} Y' \xleftarrow{\phi_Y} Y$$
$$i \uparrow \qquad \uparrow i' \qquad \uparrow i$$
$$X \xleftarrow{\varphi_X} X' \xleftarrow{\phi_X} X$$

commutes. In this situation, if i' is doubly projective, then i is also doubly projective.

Proof. Consider a surjective unital $*$-homomorphism $p\colon A \to B$ and two unital $*$-homomorphisms $f\colon X \to A$ and $g\colon Y \to B$ such that $g \circ i = p \circ f$. Here is the corresponding diagram

$$B \xleftarrow{g} Y \xleftarrow{\varphi_Y} Y' \xleftarrow{\phi_Y} Y$$
$$p \uparrow \qquad i \uparrow \qquad \uparrow i' \qquad \uparrow i$$
$$A \xleftarrow{f} X \xleftarrow{\varphi_X} X' \xleftarrow{\phi_X} X$$

Let $f' = f \circ \varphi_X \colon X' \to A$ and $g' = g \circ \varphi_Y \colon Y' \to B$. Note that

$$g' \circ i' = g \circ \varphi_Y \circ i' = g \circ i \circ \varphi_X = p \circ f \circ \varphi_X = p \circ f'.$$

Since i' is doubly projective, there exists a unital $*$-homomorphism $h'\colon Y' \to A$ such that $p \circ h' = g'$ and $h' \circ i' = f'$.

Now consider the composition $h = h' \circ \phi_Y \colon Y \to B$ and observe that

$$p \circ h = p \circ h' \circ \phi_Y = g' \circ \phi_Y = g \circ \varphi_Y \circ \phi_Y = g$$

and

$$h \circ i = h' \circ \phi_Y \circ i = h' \circ i' \circ \phi_X = f' \circ \phi_X = f \circ \varphi_X \circ \phi_X = f.$$

This shows that i is doubly projective. $\qquad\square$

The following statement provides an important class of doubly projective homomorphisms.

Lemma 5.7. *Let X be a unital C^*-algebra and Y be a projective unital C^*-algebra. Then the canonical inclusion $\pi_X\colon X \hookrightarrow X \bigstar_{\mathbb{C}} Y$ is doubly projective.*

Proof. Consider a surjective unital $*$-homomorphism $p\colon A \to B$ and two unital $*$-homomorphisms $f\colon X \to A$ and $g\colon X \bigstar_{\mathbb{C}} Y \to B$ such that $p \circ f = g \circ i_X$. Our goal is to construct a unital $*$-homomorphism $h\colon X \bigstar_{\mathbb{C}} Y \to A$ such that $p \circ h = g$ and $h \circ \pi_X = f$. Let $\pi_Y\colon Y \to X \bigstar_{\mathbb{C}} Y$ denote the canonical embedding of Y into $X \bigstar_{\mathbb{C}} Y$. Since Y is projective, there exists a unital $*$-homomorphism $h_1\colon Y \to A$

such that $p \circ h_1 = g \circ i_Y$. The two unital $*$-homomorphisms $f \colon X \to A$ and $h_1 \colon Y \to A$ define the unique unital $*$-homomorphism $h \colon X \bigstar_{\mathbb{C}} Y \to A$ such that $h \circ \pi_X = f$ and $h \circ \pi_Y = h_1$. Finally, observe that $g \circ \pi_X = p \circ f = (p \circ h) \circ \pi_X$ and $g \circ \pi_Y = p \circ h_1 = (p \circ h) \circ \pi_Y$. This shows that $p \circ h = g$. $\square$

Next we introduce the concept of a doubly projective homomorphism of separable type.

Definition 5.8. We say that a doubly projective homomorphism $i \colon X \to Y$ between projective unital C^*-algebras has a separable type, if there exist a projective unital C^*-algebra X' such that $d(X') = d(X)$, a separable projective unital C^*-algebra Y' and two surjective unital $*$-homomorphisms $\varphi_X \colon X' \to X$ and $\varphi_Y \colon X' \bigstar_{\mathbb{C}} Y' \to Y$ such that $i \circ \varphi_X = \varphi_Y \circ \pi_{X'}$, where $\pi_{X'} \colon X' \to X' \bigstar_{\mathbb{C}} Y'$ denotes the natural inclusion. In other words we require the commutativity of the following diagram

$$
\begin{array}{ccc}
Y & \xleftarrow{\ \varphi_Y\ } & X' \bigstar_{\mathbb{C}} Y' \\
\Big\uparrow{\scriptstyle i} & & \Big\uparrow{\scriptstyle \pi_{X'}} \\
X & \xleftarrow{\ \varphi_X\ } & X'.
\end{array}
$$

Lemma 5.9. *Every doubly projective homomorphism between separable projective unital C^*-algebras has a separable type.*

Proof. Let $i \colon X \to Y$ be a doubly projective homomorphism and X and Y be separable projective unital C^*-algebras. Consider the unital free product $X \bigstar_{\mathbb{C}} Y$ and note that the diagram

$$
\begin{array}{ccc}
Y & \xleftarrow{\ i \bigstar_{\mathbb{C}} \mathrm{id}_Y\ } & X \bigstar_{\mathbb{C}} Y \\
\Big\uparrow{\scriptstyle i} & & \Big\uparrow{\scriptstyle \pi_X} \\
X & \xleftarrow{\ \mathrm{id}_X\ } & X
\end{array}
$$

commutes. Also observe that $X \bigstar_{\mathbb{C}} Y$ is a projective unital C^*-algebra. Clearly $i \bigstar_{\mathbb{C}} \mathrm{id}_Y \colon X \bigstar_{\mathbb{C}} Y \to X$ is surjective, because $(i \bigstar_{\mathbb{C}} \mathrm{id}_Y) \circ \pi_Y = \mathrm{id}_Y$ is surjective. $\square$

Lemma 5.10. *Let $\mathcal{S} = \{X_\alpha, i_\alpha^{\alpha+1}, \tau\}$ be a well-ordered continuous direct system of unital C^*-algebras. If the short injection $i_\alpha^{\alpha+1} \colon X_\alpha \to X_{\alpha+1}$ of the system $\mathcal{S}$ is doubly projective for each $\alpha < \tau$, then the limit injection $i_0 \colon X_0 \to \varinjlim \mathcal{S}$ is also doubly projective.*

Proof. Let $p \colon A \to B$ be a surjective unital $*$-homomorphism of unital C^*-algebras. Let also $g \colon X_0 \to A$ and $h \colon \varinjlim \mathcal{S}_X \to B$ be unital $*$-homomorphisms such that $p \circ g = h \circ i_0$.

By induction we construct a well ordered collection $\{k_\alpha \colon X_\alpha \to A; \alpha < \tau\}$ of unital $*$-homomorphisms. Let $k_0 = g$ and suppose that we have already constructed $*$-homomorphisms k_α for each $\alpha < \gamma$, where $\gamma < \tau$, in such a way that the following conditions are satisfied:

(a) $k_\alpha = k_{\alpha+1} \circ i_\alpha^{\alpha+1}$ for each $\alpha < \gamma$.
(b) $p \circ k_\alpha = h \circ i_\alpha$ for each $\alpha < \tau$.
(c) $k_\beta = \varinjlim\{k_\alpha; \alpha < \beta\}$ whenever β is a limit ordinal number with $\beta < \gamma$.

Let us construct a $*$-homomorphism $k_\gamma \colon X_\gamma \to A$.
If γ is a limit ordinal number, then let $k_\gamma = \varinjlim\{k_\alpha; \alpha < \gamma\}$.
If $\gamma = \alpha + 1$, then consider the following commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\ p\ } & B \\[2mm]
k_\alpha \uparrow & & \uparrow h \circ i_\alpha \\[2mm]
X_\alpha & \xrightarrow{\ i_\alpha^{\alpha+1}\ } & X_{\alpha+1}
\end{array}
$$

Since $i_\alpha^{\alpha+1} \colon X_\alpha \to X_{\alpha+1}$ is doubly projective there exists a unital $*$-homomorphism $k_{\alpha+1} \colon X_{\alpha+1} \to A$ such that $k_\alpha = k_{\alpha+1} \circ i\alpha^{\alpha+1}$ and $p \circ k_{\alpha+1} = h \circ i_\alpha$.

Thus, the homomorphisms $k_\alpha \colon X_\alpha \to A$ are constructed for each $\alpha < \tau$ and satisfy the above stated properties for each $\alpha < \tau$. It only remains to note that for the unital $*$-homomorphism $k = \varinjlim\{k_\alpha; \alpha < \tau\} \colon \varinjlim \mathcal{S}_X \to A$ we have $g = k_0 = k \circ i_0$ and $h = p \circ k$ as required. $\qquad\square$

As was pointed out in the introduction, there is a deeper relation between doubly projective homomorphisms and projective C^*-algebras, than it might appear to be the case. Let $\mathrm{Mor}(\mathcal{C}_1^*)$ denote the category of unital $*$-homomorphisms of unital C^*-algebras. The following statement is true.

Proposition 5.11. *The following conditions are equivalent for a unital $*$-homomorphism $f \colon X \to Y$ of projective unital C^*-algebras:*

(a) *f is doubly projective.*
(b) *f is a projective object of the category $\mathrm{Mor}(\mathcal{C}_1^*)$.*

Proof. (a) $\Longrightarrow$ (b). Let $p \colon A \to B$ and $q \colon C \to D$ be objects of the category $\mathrm{Mor}(\mathcal{C}_1^*)$ and $(s, r) \colon p \to q$ be an epimorphism of the same category. Our goal is to show that for any morphism $(\alpha, \beta) \colon f \to q$ of $\mathrm{Mor}(\mathcal{C}_1^*)$ there exists a morphism $(\widetilde{\alpha}, \widetilde{\beta}) \colon f \to p$ of $\mathrm{Mor}(\mathcal{C}_1^*)$ such that $(s, r) \circ (\widetilde{\alpha}, \widetilde{\beta}) = (\alpha, \beta)$.
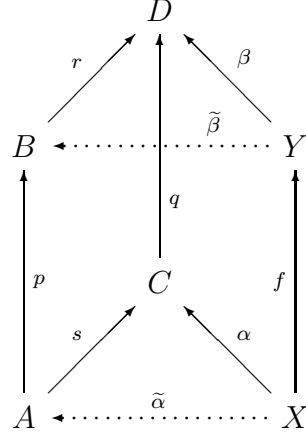
Since (r, s) is an epimorphism in $\mathrm{Mor}(\mathcal{C}_1^*)$ it follows that each of the homo-morphisms r and s is surjective. Since X is projective, there exists a unital $*$-homomorphism $\widetilde{\alpha}\colon X \to A$ such that $s \circ \widetilde{\alpha} = \alpha$. Clearly $r \circ p \circ \widetilde{\alpha} = q \circ s \circ \widetilde{\alpha} = q \circ \alpha = \beta \circ f$. Consequently, since f is doubly projective, there exists a unital $*$-homomorphism $\widetilde{\beta}\colon Y \to B$ such that $\widetilde{\beta} \circ f = p \circ \widetilde{\alpha}$ and $r \circ \widetilde{\beta} = \beta$. In other words the following diagram
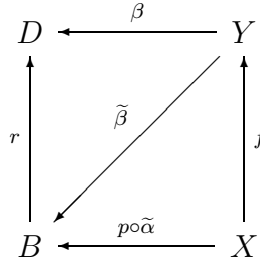
commutes. The straitforward verification shows that $(s, r) \circ (\widetilde{\alpha}, \widetilde{\beta}) = (\alpha, \beta)$ as required.

(b) $\Longrightarrow$ (a). Now suppose that $f\colon X \to Y$ is a projective object of the cate-gory $\mathrm{Mor}(\mathcal{C}_1^*)$. In order to show that f is doubly projective, consider a surjective unital $*$-homomorphism $p\colon A \to B$ and two unital $*$-homomorphisms $g\colon X \to A$ and $h\colon Y \to B$ such that $p \circ g = h \circ f$. Clearly the pair (g, h) forms a mor-phism $(g, h)\colon f \to p$ in the category $\mathrm{Mor}(\mathcal{C}_1^*)$. Consider also the epimorphism (in the category $\mathrm{Mor}(\mathcal{C}_1^*)$) $(\mathrm{id}_A, p)\colon \mathrm{id}_A \to p$ sending the left vertical arrow in the following diagram onto the middle one.

$$A \xrightarrow{\ p\ } B \xleftarrow{\ h\ } Y$$

$$\mathrm{id}_A \uparrow \qquad\qquad \uparrow p \qquad\qquad \uparrow f$$

$$A \xrightarrow{\ \mathrm{id}_A\ } A \xleftarrow{\ g\ } X$$

Since f is a projective object in the category $\mathrm{Mor}(\mathcal{C}_1^*)$, it follows that there exists a morphism $(\widetilde{g}, \widetilde{h}) \colon f \to \mathrm{id}_A$, consisting of the unital $*$-homomorphisms $\widetilde{g} \colon X \to A$ and $\widetilde{h} \colon Y \to A$, such that $(\mathrm{id}_A, p) \circ (\widetilde{g}, \widetilde{h}) = (g, h)$. This implies that $\mathrm{id}_A \circ \widetilde{g} = g$, i.e. $\widetilde{g} = g$, and $p \circ \widetilde{h} = h$. In order to prove the equality $\widetilde{h} \circ f = g$, simply note that $(\widetilde{g}, \widetilde{h}) \colon f \to \mathrm{id}_A$ is a morphism in the category $\mathrm{Mor}(\mathcal{C}_1^*)$. Thus f is doubly projective. □

5.2. **Characterization of projective unital C^*-algebras.** We begin with the following preliminary result.

Lemma 5.12. *Let X be a projective unital C^*-algebra of density $\tau > \omega$. Then X admits a direct C_ω^*-system $\mathcal{S}_X = \{X_t, i_s^t, \mathcal{A}\}$, consisting of separable projective unital C^*-subalgebras of X. We may assume that $\mathcal{A}$ is a cofinal and ω-closed subset of $\exp_\omega \tau$.*

Proof. Let A be a dense subset of X such that $|A| = \tau$. Let also $T = \exp_\omega A$. Since $\tau > \omega$, it follows that $|T| = \tau$. As in the proof of Proposition 3.2, we can conclude that X is the limit of the direct system $\{X_t, i_s^t, T\}$, consisting of separable unital C^*-subalgebras of X (generated by countable subsets of A) and associated inclusion maps.

Next consider the unital $*$-homomorphism $\varphi \colon \bigstar_\mathbb{C}\{X_t \colon t \in T\} \to X$, generated by the homomorphisms $i_t \colon X_t \to X$. This means that $\varphi \circ \pi_{X_t} = i_t$ for each $t \in T$ (here $\pi_{X_t} \colon X_t \to \bigstar_\mathbb{C}\{X_t \colon t \in T\}$ denotes the canonical inclusion). Note that φ is a surjective unital $*$-homomorphism. This follows from Lemma 3.3.

Recall that by Proposition 4.4, the collection

$$\mathcal{S} = \left\{ \bigstar_\mathbb{C}\{X_t \colon t \in S\}, \pi_S^R, \exp_\omega T \right\}$$

is a direct C_ω^*-system such that $\bigstar_\mathbb{C}\{X_t \colon t \in T\} = \varinjlim \mathcal{S}$.

For each $S \subseteq T$ let $X_S = \mathrm{cl}_X \varphi\left(\bigstar_\mathbb{C}\{X_t \colon t \in S\}\right)$. Also by $\varphi_S \colon \bigstar_\mathbb{C}\{X_t \colon t \in S\} \to X_S$ we denote the restriction of the homomorphism φ onto the unital free product $\bigstar_\mathbb{C}\{X_t \colon t \in S\}$. We have the following commutative diagram

$$X \xleftarrow{\ \varphi\ } \bigstar_\mathbb{C}\{X_t \colon t \in T\}$$

$$i_S \uparrow \qquad\qquad\qquad \uparrow \pi_S^T$$

$$X_S \xleftarrow{\ \varphi_S\ } \bigstar_\mathbb{C}\{X_t \colon t \in S\},$$

where $i_S \colon X_S \to X$ denotes the inclusion.

It is obvious that the system $\mathcal{S}_X = \{X_S, i_S^R, \exp_\omega T\}$, consisting of C^*-subalgebras X_S of X and their natural inclusions $i_S^R \colon X_S \to X_R$, forms a direct C_ω^*-system such that $X = \varinjlim \mathcal{S}_X$. Also note that

$$\{\varphi_S \colon S \in \exp_\omega T\} \colon \mathcal{S} \to \mathcal{S}_X$$

is a morphism between the indicated direct systems such that $\varphi = \varinjlim \{\varphi_S \colon S \in \exp_\omega T\}$.

Since X is a projective C^*-algebra, there exists a unital $*$-homomorphism $\phi \colon X \to \bigstar_\mathbb{C}\{X_t \colon t \in T\}$ such that $\varphi \circ \phi = \mathrm{id}_X$.

According to Theorem 3.5, applied to the homomorphism $\phi \colon \varinjlim \mathcal{S}_X \to \varinjlim \mathcal{S}$, there exist a cofinal and ω-closed subset $\mathcal{A}$ of $\exp_\omega \tau$ and a morphism

$$\{\phi_S \colon S \in \mathcal{A}\} \colon \mathcal{S}_X | \mathcal{A} \to \mathcal{S} | \mathcal{A}$$

such that $\phi = \varinjlim \{\phi_S \colon S \in \mathcal{A}\}$. In particular, the square diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ \phi\ } & \bigstar_\mathbb{C}\{X_t \colon t \in T\} \\
{\scriptstyle i_S}\big\uparrow & & \big\uparrow{\scriptstyle \pi_S^T} \\
X_S & \xrightarrow{\ \phi_S\ } & \bigstar_\mathbb{C}\{X_t \colon t \in S\},
\end{array}
$$

commutes for each $S \in \mathcal{A}$. Note also that $\varphi_S \circ \phi_S = \mathrm{id}_{X_S}$ for each $S \in \mathcal{A}$.

According to Lemma 4.6, the C^*-algebra X, and hence each X_t, $t \in T$, admits a unital $*$-homomorphism into $\mathbb{C}$. Consequently, by Remark 4.5, the inclusion $\pi_S^T \colon \bigstar_\mathbb{C}\{X_t \colon t \in S\} \to \bigstar_\mathbb{C}\{X_t \colon t \in T\}$ is a coretraction with the associated retraction $g_S^T \colon \bigstar_\mathbb{C}\{X_t \colon t \in T\} \to \bigstar_\mathbb{C}\{X_t \colon t \in S\}$. Consider the unital $*$-homomorphism $r_S \colon X \to X_S$, defined as the composition $r_S = \varphi_S \circ g_S^T \circ \phi$. Note that

$$r_S \circ i_S = \varphi_S \circ g_S^T \circ \phi \circ i_S = \varphi_S \circ g_S^T \circ \pi_S^T \circ \phi_S = \varphi_S \circ \phi_S = \mathrm{id}_{X_S},$$

which shows that r_S is a retraction. It only remains to note that X_S, as a retract of X, is projective. $\qquad\square$

The following statement provides a characterization of non-separable projective unital C^*-algebras. It should be noted that condition (b) of Theorem 5.13 is significantly stronger than the conclusion of Lemma 5.12.

Theorem 5.13. *The following conditions are equivalent for any unital C^*-algebra X of density $d(X) = \tau > \omega$:*

(a) *X is projective.*
(b) *X is isomorphic to the limit of a direct C_ω^*-system $\mathcal{S}_X = \{X_\alpha, i_\alpha^\beta, \mathcal{A}\}$, consisting of separable projective unital C^*-algebras X_α and doubly projective*

limit injections $i_\alpha\colon X_\alpha \to X$, $\alpha \in \mathcal{A}$. We may assume that $\mathcal{A}$ is cofinal and ω-closed in $\exp_\omega \tau$.

(c) *X is isomorphic to the limit of a well-ordered continuous direct system $\mathcal{S}_X = \{X_\alpha, i_\alpha^{\alpha+1}, \tau\}$ of length τ satisfying the following properties:*

1. *X_α is a projective unital C^*-algebra for each $\alpha < \tau$.*
2. *Short injection $i_\alpha^{\alpha+1}\colon X_\alpha \to X_{\alpha+1}$ is doubly projective and has a separable type for each $\alpha < \tau$.*
3. *X_0 is a separable projective unital C^*-algebra.*

Proof. Part I. First we show that if X is a projective C^*-algebra, then there exists a well ordered continuous direct system $\mathcal{S}_X = \{X_\alpha, i_\alpha^{\alpha+1}, \tau\}$, satisfying condition (c). While proving this we will show the existence of a direct C_ω^*-system $\{X_\alpha, i_\alpha^\beta, \mathcal{A}\}$, satisfying condition (b).

According to condition (a) and Lemmas 5.12 and 3.3 there exists a collection $\{X_t\colon t \in T\}$, consisting of separable unital projective C^*-subalgebra of X, such that $X = \cup\{X_t\colon t \in T\}$ and $|T| = \tau$.

Below we follow the proof of Lemma 5.12. The fact that each X_t, $t \in \mathcal{A}$, is projective becomes crucial later in this proof.

As in the proof of Lemma 5.12, the homomorphisms $i_t\colon X_t \to X$, $t \in T$, generate the surjective unital $*$-homomorphism $\varphi\colon \bigstar_\mathbb{C}\{X_t\colon t \in T\} \to X$ such that $\varphi \circ \pi_{X_t} = i_t$ for each $t \in T$ (here $\pi_{X_t}\colon X_t \to \bigstar_\mathbb{C}\{X_t\colon t \in T\}$ denotes the canonical inclusion).

Recall that by Proposition 4.4, the collection

$$\mathcal{S} = \left\{\bigstar_\mathbb{C}\{X_t\colon t \in S\}, \pi_S^R, \exp_\omega T\right\}$$

is a direct C_ω^*-system such that $\bigstar_\mathbb{C}\{X_t\colon t \in T\} = \varinjlim \mathcal{S}$.

For each $S \subseteq T$ let $X_S = \mathrm{cl}_X \varphi\left(\bigstar_\mathbb{C}\{X_t\colon t \in S\}\right)$. Also by $\varphi_S\colon \bigstar_\mathbb{C}\{X_t\colon t \in S\} \to X_S$ we denote the restriction of the homomorphism φ onto the unital free product $\bigstar_\mathbb{C}\{X_t\colon t \in S\}$. We have the following commutative diagram

$$
\begin{array}{ccc}
X & \xleftarrow{\ \varphi\ } & \bigstar_\mathbb{C}\{X_t\colon t \in T\} \\
{\scriptstyle i_S}\big\uparrow & & \big\uparrow{\scriptstyle \pi_S^T} \\
X_S & \xleftarrow{\ \varphi_S\ } & \bigstar_\mathbb{C}\{X_t\colon t \in S\},
\end{array}
$$

where $i_S\colon X_S \to X$ denotes the inclusion.

It is obvious that the system $\mathcal{S}_X = \{X_S, i_S^R, \exp_\omega T\}$, consisting of C^*-subalgebras X_S of X and their natural inclusions $i_S^R\colon X_S \to X_R$, forms a direct C_ω^*-system such that $X = \varinjlim \mathcal{S}_X$.

Since, by (a), X is a projective C^*-algebra, there exists a unital $*$-homomorphism $\phi\colon X \to \bigstar_\mathbb{C}\{X_t\colon t \in T\}$ such that $\varphi \circ \phi = \mathrm{id}_X$.

Let us say that a subset $S \subseteq T$ is admissible if $\phi(X_S) \subseteq \bigstar_{\mathbb{C}}\{X_t : t \in S\}$. This clearly means that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ \phi\ } & \bigstar_{\mathbb{C}}\{X_t : t \in T\} \\[4pt]
{\scriptstyle i_S}\big\uparrow & & \big\uparrow{\scriptstyle \pi_S^T} \\[4pt]
X_S & \xrightarrow{\ \phi_S\ } & \bigstar_{\mathbb{C}}\{X_t : t \in S\},
\end{array}
$$

where $\phi_S = \phi|X_S : X_S \to \bigstar_{\mathbb{C}}\{X_t : t \in S\}$, commutes.

We need to state some of the properties of admissible subsets.

Claim 1. *If $S \subseteq T$ is admissible, then $\phi_S \circ \varphi_S = \mathrm{id}_{X_S}$.*
Proof of Claim 1. Follows form the above constructions and the equality $\phi \circ \varphi = \mathrm{id}_X$ (see the proof of Lemma 5.12).

Claim 2. *If S is an admissible subset of T, then $X_S = \varphi\left(\bigstar_{\mathbb{C}}\{X_t : t \in S\}\right)$.*
Proof of Claim 2. Follows from Claim 1 (see the proof of Lemma 5.12).

Claim 3. *The union of an arbitrary collection of admissible subsets of T is admissible.*
Proof of Claim 3. Let S_i, $i \in I$ be an admissible subset of T and let $S = \bigcup\{S_i : i \in I\}$. First observe that

$$
X_S = \mathrm{cl}_X\,\varphi\left(\bigstar_{\mathbb{C}}\{X_t : t \in S\}\right) = \mathrm{cl}_X\,\varphi\left(\bigcup\{\bigstar_{\mathbb{C}}\{X_t : t \in S_i\} : i \in I\}\right) =
$$
$$
\mathrm{cl}_X\left(\bigcup\{\varphi\left(\{X_t : t \in S_i\}\right) : i \in I\right) \subseteq \mathrm{cl}_X\left(\bigcup\{X_{S_i} : i \in I\}\right).
$$

Consequently

$$
\phi(X_S) \subseteq \phi\left(\mathrm{cl}_X\left(\bigcup\{X_{S_i} : i \in I\}\right)\right) \subseteq \mathrm{cl}_{\bigstar_{\mathbb{C}}\{X_t : t \in T\}}\left(\phi\left(\bigcup\{X_{S_i} : i \in I\}\right)\right) \subseteq
$$
$$
\mathrm{cl}_{\bigstar_{\mathbb{C}}\{X_t : t \in T\}}\left(\bigcup\{\phi(X_{S_i}) : i \in I\}\right) \subseteq \mathrm{cl}_{\bigstar_{\mathbb{C}}\{X_t : t \in T\}}\left(\bigcup\{\bigstar_{\mathbb{C}}\{X_t : t \in S_i\} : i \in I\}\right)
$$
$$
\subseteq \bigstar_{\mathbb{C}}\{X_t : t \in S\}.
$$

Claim 4. *If S is an admissible subset of T, then X_S is a projective C^*-subalgebra of X.*
Proof of Claim 4. See the proof of lemma 5.12.

Claim 5. *Every countable subset of T is contained in a countable admissible subset of T.*
Proof of Claim 5. According to Theorem 3.5, applied to the homomorphism $\phi \colon \varinjlim \mathcal{S}_X \to \varinjlim \mathcal{S}$, there exist a cofinal and ω-closed subset $\mathcal{A}$ of $\exp_\omega \tau$ and a morphism

$$
\{\phi_S \colon S \in \mathcal{A}\} \colon \mathcal{S}_X|\mathcal{A} \to \mathcal{S}|\mathcal{A}
$$

such that $\phi = \varinjlim\{\phi_S\colon S \in \mathcal{A}\}$. Clearly each $S \in \mathcal{A}$ is admissible.

Claim 6. *If S is an admissible subset of T, then the inclusion $i_S\colon X_S \to X$ is doubly projective.*

Proof of Claim 6. Recall that the following diagram

$$
\begin{array}{ccccc}
X & \xleftarrow{\;\varphi\;} & \bigstar_{\mathbb{C}}\{X_t\colon t \in T\} & \xleftarrow{\;\phi\;} & X \\[2pt]
{\scriptstyle i_S}\big\uparrow & & \big\uparrow{\scriptstyle \pi_S^T} & & \big\uparrow{\scriptstyle i_S} \\[2pt]
X_S & \xleftarrow{\;\varphi_S\;} & \bigstar_{\mathbb{C}}\{X_t\colon t \in S\} & \xleftarrow{\;\phi_S\;} & X_S
\end{array}
$$

commutes and that $\varphi \circ \phi = \mathrm{id}_X$ and $\varphi_S \circ \phi_S = \mathrm{id}_{X_S}$.

Since each X_t, $t \in T$, is projective (this is where Lemma 5.12 is actually being used) we easily conclude that $\bigstar_{\mathbb{C}}\{X_t\colon t \in T - S\}$ is also projective (compare to [2, Propositions 2.31, 2.32]). By Lemmas 4.1 and 5.7, the inclusion $\pi_S^T\colon \bigstar_{\mathbb{C}}\{X_t\colon t \in S\} \to \bigstar_{\mathbb{C}}\{X_t\colon t \in T\}$ is doubly projective. Finally, Lemma 5.6 guarantees that the inclusion $i_S\colon X_S \to X$ is also doubly projective. This completes proof of Claim 6.

Now consider the direct system $\mathcal{S}_X|\mathcal{A} = \{X_S, i_S^R, \mathcal{A}\}$. Clearly $\mathcal{S}_X|\mathcal{A}$ is a direct C_ω^*-system such that $X = \varinjlim \mathcal{S}_X|\mathcal{A}$ (see Claim 5). By Claim 4, each X_S, $S \in \mathcal{A}$, is a separable unital projective subalgebra of X and, by Claim 6, each limit inclusion $i_S\colon X_S \to X$, $S \in \mathcal{A}$, is doubly projective. This finishes the proof of the implication (a) $\Longrightarrow$ (b).

Next we prove the implication (a) $\Longrightarrow$ (c). Since $|T| = \tau$, we can write $T = \{t_\alpha\colon \alpha < \tau\}$. By Claim 5, for each $\alpha < \tau$ there exists a countable admissible subset $S_\alpha \subseteq T$ such that $t_\alpha \in S_\alpha$. Let $T_\alpha = \cup\{S_\alpha\colon \beta \le \alpha\}$ and $X_\alpha = X_{T_\alpha}$. Also let $i_\alpha^{\alpha+1}\colon X_\alpha \to X_{\alpha+1}$ denote the inclusion. Thus we have the well ordered continuous direct system $\mathcal{S}_X = \{X_\alpha, i_\alpha^{\alpha+1}, \tau\}$. It follows from the above constructions that $X = \varinjlim \mathcal{S}_X$. According to Claims 3 and 4, each X_α, $\alpha < \tau$, is a unital projective C^*-subalgebra of X. Since $T_0 = S_0$ is countable, we conclude that X_0 is separable. Claim 6 guarantees that for each $\alpha < \tau$, both limit inclusions $i_\alpha\colon X_\alpha \to X$ and $i_{\alpha+1}\colon X_{\alpha+1} \to X$ are doubly projective. Note that $i_\alpha = i_{\alpha+1} \circ i_\alpha^{\alpha+1}$. By Lemma 5.2, $i_{\alpha+1}$ is a coretraction. Consequently, by Lemma 5.5, $i_\alpha^{\alpha+1}$ is also doubly projective. Finally, in order to see that $i_\alpha^{\alpha+1}$ has a separable type, note that according to the above constructions and Lemma 4.1, we have the following commuting diagram

$$
\begin{array}{ccc}
X_{\alpha+1} & \xleftarrow{\;\varphi_{T_{\alpha+1}}\;} & (\bigstar_{\mathbb{C}}\{X_t\colon t \in T_\alpha\})\,\bigstar_{\mathbb{C}}\,(\bigstar_{\mathbb{C}}\{X_t\colon t \in S_{\alpha+1}\}) \\[2pt]
{\scriptstyle i_S}\big\uparrow & & \big\uparrow{\scriptstyle \pi_{\bigstar_{\mathbb{C}}\{X_t\colon\, t\in T_\alpha\}}} \\[2pt]
X_\alpha & \xleftarrow{\;\varphi_{T_\alpha}\;} & \bigstar_{\mathbb{C}}\{X_t\colon t \in T_\alpha\},
\end{array}
$$

with surjective φ_{T_α} and $\varphi_{T_{\alpha+1}}$. Clearly both

$$\bigstar_{\mathbb{C}}\{X_t \colon t \in T_\alpha\} \ \text{and} \ \left(\bigstar_{\mathbb{C}}\{X_t \colon t \in T_\alpha\}\right) \bigstar_{\mathbb{C}} \left(\bigstar_{\mathbb{C}}\{X_t \colon t \in S_{\alpha+1}\}\right)$$

are projective (as unital free products of projective C^*-algebras). It only remains to note that since $S_{\alpha+1}$ is countable and since each X_t is separable, the unital free product $\bigstar_{\mathbb{C}}\{X_t \colon t \in S_{\alpha+1}\}$ is also separable ([1, Theorem 3.1]). This completes the proof of the implication (a) $\Longrightarrow$ (c).

In order to prove the implication (b) $\Longrightarrow$ (a) observe that if $\mathcal{S}_X = \{X_\alpha, i_\alpha^\beta, \mathcal{A}\}$ is a direct C_ω^*-system satisfying properties indicated in condition (b), then for any $\alpha \in \mathcal{A}$ the α-th limit inclusion $i_\alpha \colon X_\alpha \to X$ is doubly projective and the C^*-algebra X_α is projective. Consequently, by lemma 5.3, X is also projective.

Finally, the implication (c) $\Longrightarrow$ (a) follows from Lemmas 5.10 and 5.3. $\square$

6. Basic properties of doubly projective square diagrams and characterization of doubly projective homomorphisms

6.1. **Doubly projective diagrams.** The pushout construction [8] applied to the category $\mathcal{C}_1^*$ leads us to the following definition [7], [10]. A commutative square diagram $X_1 X_2 Y_2 Y_1$, consisting of unital C^*-algebras and unital $*$-homomorphisms, is called *pushout*, if for any two coherent unital $*$-homomorphisms $g \colon X_2 \to Z$ and $h \colon Y_1 \to Z$ into any unital C^*-algebra Z (i.e. $g \circ i = h \circ f_1$), there exists unique unital $*$-homomorphism $g \bigstar h \colon Y_2 \to Z$ (a more informative notation $g \bigstar_{X_1} h$ for the sake of simplicity is replaced by $g \bigstar h$) such that $(g \bigstar h) \circ f_2 = g$ and $(g \bigstar h) \circ j = h$:
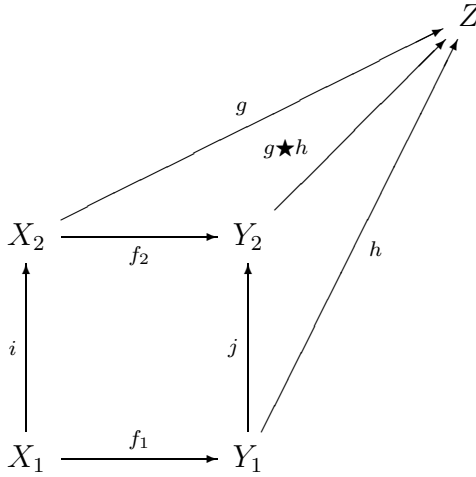
C^*-algebra Y_2 in such a case is isomorphic to the amalgamated free product $X_2 \bigstar_{X_1} Y_1$, which is the quotient of the unital free product $X_2 \bigstar_{\mathbb{C}} Y_1$ by the

closed ideal generated by $\{i(x) - f_1(x) \colon x \in X_1\}$. Also if $\pi \colon X_2 \bigstar_{\mathbb{C}} Y_1 \to Y_2$ denotes this quotient homomorphism, then $\pi \circ \pi_{Y_1} = j$ and $\pi \circ \pi_{X_2} = f_2$, where $\pi_{Y_1} \colon Y_1 \to X_2 \bigstar_{\mathbb{C}} Y_1$ and $\pi_{X_2} \colon X_2 \to X_2 \bigstar_{\mathbb{C}} Y_1$ denote canonical embeddings. Here is the corresponding diagram

Lemma 6.1. *Let*

$$
\begin{array}{ccc}
X_2 & \xrightarrow{\;f_2\;} & Y_2 \\
i\uparrow & & \uparrow j \\
X_1 & \xrightarrow{\;f_1\;} & Y_1
\end{array}
$$

be a pushout diagram, consisting of unital C^*-algebras and unital $*$-homomorphisms. If f_1 is doubly projective, then f_2 is also doubly projective.

Proof. Let $p \colon A \to B$ be a surjective unital $*$-homomorphism of unital C^*-algebras. Consider also two unital $*$-homomorphisms $g \colon X_2 \to A$ and $h \colon Y_2 \to B$ such that $p \circ g = h \circ f_2$. Clearly

$$
p \circ (g \circ i) = (p \circ g) \circ i = (h \circ f_2) \circ i = (h \circ j) \circ f_1 = h \circ (j \circ f_1).
$$

Since f_1 is doubly projective, there exists a unital $*$-homomorphism $\widetilde{k} \colon Y_2 \to A$ such that $g \circ i = \widetilde{k} \circ f_1$ and $h \circ j = p \circ \widetilde{k}$. Since the given diagram is a pushout, we have a unital $*$-homomorphism $k = g \bigstar \widetilde{k} \colon Y_2 \to A$. Recall that $g = (g \bigstar \widetilde{k}) \circ f_2$ and $\widetilde{k} = (g \bigstar \widetilde{k}) \circ j$. Consequently it only remains to show that $p \circ (g \bigstar \widetilde{k}) = h$. In order to prove this equality note that

$$
\left[p \circ (g \bigstar \widetilde{k}) \right] \circ f_2 = p \circ g = h \circ f_2 \;\; \text{and} \;\; \left[p \circ (g \bigstar \widetilde{k}) \right] \circ j = p \circ \widetilde{k} = h \circ j.
$$

Again, since the given diagram is a pushout, the above equalities imply that $p \circ (g \bigstar \widetilde{k}) = h$ as required. $\qquad\square$

Lemma 6.2. *Let $f\colon X \to Y$ be a unital $*$-homomorphism of unital C^*-algebras. Let also A be a unital C^*-algebra. Then the diagram*

$$
\begin{array}{ccc}
Y & \xrightarrow{\ \pi_Y\ } & Y\bigstar_{\mathbb{C}}A \\[4pt]
f\big\uparrow & & \big\uparrow {\scriptstyle(\pi_Y\circ f)\bigstar \,\mathrm{id}_A} \\[4pt]
X & \xrightarrow{\ \pi_X\ } & X\bigstar_{\mathbb{C}}A
\end{array}
$$

is a pushout.

Proof. Consider the pushout

$$
\begin{array}{ccc}
Y & \xrightarrow{\ \varphi\ } & Z \\[4pt]
f\big\uparrow & & \big\uparrow {\scriptstyle\phi} \\[4pt]
X & \xrightarrow{\ \pi_X\ } & X\bigstar_{\mathbb{C}}A
\end{array}
$$

generated by the homomorphisms $f\colon X \to Y$ and $\pi_X\colon X \to X\bigstar_{\mathbb{C}}A$. Since, by the commutativity of the first diagram,

$$(6.1) \qquad\qquad \pi_Y \circ f = \big[(\pi_Y \circ f)\bigstar \mathrm{id}_A\big] \circ \pi_X \ ,$$

it follows that there exists unique unital $*$-homomorphism $p\colon Z \to Y\bigstar_{\mathbb{C}}A$ such that

$$(6.2) \qquad\qquad\qquad\qquad p \circ \varphi = \pi_Y$$

and

$$(6.3) \qquad\qquad\qquad\qquad p \circ \phi = (\pi_Y \circ f)\bigstar \mathrm{id}_A \ .$$

Let $\pi_A\colon A \to X\bigstar_{\mathbb{C}}A$ denote the canonical injection of A into the unital free product $X\bigstar_{\mathbb{C}}A$. Consider the homomorphisms $\varphi\colon Y \to Z$ and $\phi \circ \pi_A\colon A \to Z$. Since $Y\bigstar_{\mathbb{C}}A$ is the unital free product, there exists unique unital $*$-homomorphism $q\colon Y\bigstar_{\mathbb{C}}A \to Z$ such that

$$(6.4) \qquad\qquad\qquad\qquad q \circ \pi_Y = \varphi$$

and

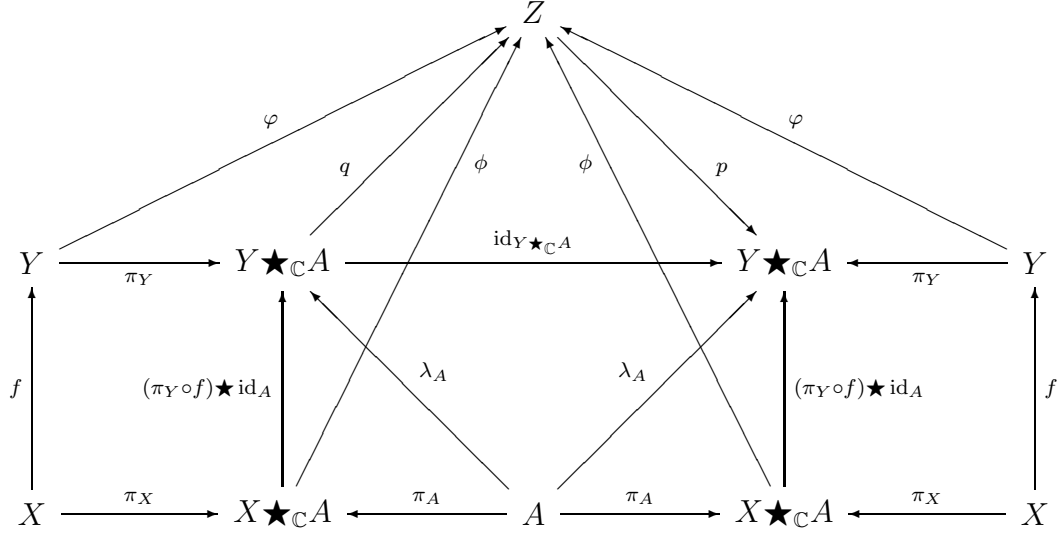$$(6.5) \qquad\qquad\qquad\qquad q \circ \lambda_A = \phi \circ \pi_A,$$

where $\lambda_A\colon A \to Y\bigstar_{\mathbb{C}}A$ denotes the canonical injection (not to be confused with π_A). Note that

$$(6.6) \qquad\qquad \lambda_A = [(\pi_Y \circ f) \bigstar \operatorname{id}_A] \circ \pi_A.$$

In order to prove our statement we need to show that p is an isomorphism. We accomplish this by proving that $q \circ p = \operatorname{id}_Z$ and $p \circ q = \operatorname{id}_{Y \bigstar_{\mathbb{C}} A}$. The following diagram helps to visualize the situation.

First let us show that

$$(6.7) \qquad\qquad q \circ [(\pi_Y \circ f) \bigstar \operatorname{id}_A] = \phi.$$

Since both $q \circ [(\pi_Y \circ f) \bigstar \operatorname{id}_A]$ and ϕ are defined on the unital free product $X \bigstar_{\mathbb{C}} A$, (6.7) will be proved by examining compositions of the above homomorphisms with π_A and π_X. Observe that

$$(6.8) \qquad \left(q \circ [(\pi_Y \circ f) \bigstar \operatorname{id}_A] \right) \circ \pi_A \overset{(6.6)}{=} q \circ \lambda_A \overset{(6.5)}{=} \phi \circ \pi_A$$

and

$$(6.9) \qquad \left(q \circ [(\pi_Y \circ f) \bigstar \operatorname{id}_A] \right) \circ \pi_X \overset{(6.1)}{=} q \circ \pi_Y \circ f \overset{(6.4)}{=} \varphi \circ f = \phi \circ \pi_X.$$

Note that (6.8) and (6.9) imply (6.7).

Next note that

$$(6.10) \qquad\qquad q \circ p \circ \phi \overset{(6.3)}{=} q \circ [(\pi_Y \circ f) \bigstar \operatorname{id}_A] \overset{(6.7)}{=} \phi$$

and

$$(6.11) \qquad\qquad q \circ p \circ \varphi \overset{(6.2)}{=} q \circ \pi_Y \overset{(6.4)}{=} \varphi.$$

Clearly (6.10) and (6.11) imply the equality $q \circ p = \mathrm{id}_Z$.

In order to establish the second equality $p \circ q = \mathrm{id}_{Y \star_{\mathbb{C}} A}$ we proceed in a similar way. Observe that

$$(6.12) \qquad p \circ q \circ \lambda_A \overset{(6.5)}{=} p \circ \phi \circ \pi_A = [(\pi_Y \circ f) \star \mathrm{id}_A] \circ \pi_A = \lambda_A$$

and

$$(6.13) \qquad\qquad p \circ q \circ \pi_Y = p \circ \varphi = \pi_Y.$$

As above, (6.12) and (6.13) imply the required equality $p \circ q = \mathrm{id}_{Y \star_{\mathbb{C}} A}$.

This shows that p is an isomorphism and completes the proof. $\qquad\square$

Lemma 6.3. *Let*

$$
\begin{array}{ccc}
X_2 & \overset{f_2}{\longrightarrow} & Y_2 \\
\;\uparrow{\scriptstyle i} & & \;\uparrow{\scriptstyle j} \\
X_1 & \overset{f_1}{\longrightarrow} & Y_1
\end{array}
$$

be a pushout diagram, consisting of projective unital C^*-algebras and doubly projective homomorphisms. If f_1 has a separable type, then f_2 also has a separable type.

Proof. Since f_1 has a separable type, we have the following commutative diagram

$$
\begin{array}{ccc}
Y_1 & \overset{\varphi_{Y_1}}{\longleftarrow} & X_1' \star_{\mathbb{C}} Y_1' \\
\;\uparrow{\scriptstyle f_1} & & \;\uparrow{\scriptstyle \pi_{X_1'}} \\
X_1 & \overset{\varphi_{X_1}}{\longleftarrow} & X_1',
\end{array}
$$

where X_1' and Y_1' are projective unital C^*-algebras, Y_1' in addition is separable and the unital $*$-homomorphisms φ_{X_1} and φ_{Y_1} are surjective. By Lemma 6.2, the diagram

$$
\begin{array}{ccc}
X_1 \star_{\mathbb{C}} Y_1' & \overset{(\pi_{X_1} \circ \varphi_{X_1}) \star_{\mathbb{C}} \mathrm{id}_{Y_1'}}{\longleftarrow} & X_1' \star_{\mathbb{C}} Y_1' \\
\;\uparrow{\scriptstyle \pi_{X_1}} & & \;\uparrow{\scriptstyle \pi_{X_1'}} \\
X_1 & \overset{\varphi_{X_1}}{\longleftarrow} & X_1',
\end{array}
$$

is a pushout. Consequently there exists a unital $*$-homomorphism $r\colon X_1\bigstar_{\mathbb{C}}Y_1' \to Y_1$ such that $\varphi_{Y_1} = r \circ \big[(\pi_{X_1}\circ\varphi_{X_1})\bigstar_{\mathbb{C}}\mathrm{id}_{Y_1'}\big]$. Since φ_{Y_1} is surjective, the latter equality guarantees that r is also surjective. Thus we have the commutative diagram

$$
\begin{array}{ccc}
Y_1 & \xleftarrow{\ \ r\ \ } & X_1\bigstar_{\mathbb{C}}Y_1' \\[2pt]
{\scriptstyle f_1}\big\uparrow & & \big\uparrow{\scriptstyle \pi_{X_1}} \\[2pt]
X_1 & \xleftarrow{\ \mathrm{id}_{X_1}\ } & X_1.
\end{array}
$$

Next consider the following diagram

in which, according to Lemma 6.2, the subdiagram, represented by the back face of the above diagram, is a pushout. Since r is surjective and since f_1 is doubly projective, there exists a unital $*$-homomorphism $s\colon Y_1 \to X_1 \to Y_1'$ such that $r\circ s = \mathrm{id}_{Y_1}$ and $\pi_{X_1} = s\circ f_1$. Now consider the unital $*$-homomorphisms $j\circ r\colon X_1\bigstar_{\mathbb{C}}Y_1' \to Y_2$ and $f_2\colon X_2 \to Y_2$. Note that $j\circ r\circ\pi_{X_1} = j\circ f_1 = f_2\circ i$. Since, as was indicated, the back face is a pushout, there exists the unique unital $*$-homomorphism $\widetilde{r}\colon X_2\bigstar_{\mathbb{C}}Y_1' \to Y_2$ such that

$$\widetilde{r}\circ\big[(\pi_{X_2}\circ i)\bigstar_{\mathbb{C}}\mathrm{id}_{Y_1'}\big] = j\circ r \ \text{ and } \ \widetilde{r}\circ\pi_{X_2} = f_2.$$

It only remains to show that $\widetilde{r}$ is surjective. To see this consider the homomorphisms $\big[(\pi_{X_2}\circ i)\bigstar_{\mathbb{C}}\mathrm{id}_{Y_1'}\big]\circ s\colon Y_1 \to X_2\bigstar_{\mathbb{C}}Y_1'$ and $\pi_{X_2}\colon X_2 \to X_2\bigstar_{\mathbb{C}}Y_1'$. Clearly

$$\big[(\pi_{X_2}\circ i)\bigstar_{\mathbb{C}}\mathrm{id}_{Y_1'}\big]\circ s\circ f_1 = \big[(\pi_{X_2}\circ i)\bigstar_{\mathbb{C}}\mathrm{id}_{Y_1'}\big]\circ\pi_{X_1} = \pi_{X_2}\circ i.$$

Since the originally given diagram is a pushout, there exists a unital $*$-homomorphism $\widetilde{s}\colon Y_2 \to X_2\bigstar_{\mathbb{C}}Y_1'$ such that $\widetilde{s}\circ j = \big[(\pi_{X_2}\circ i)\bigstar_{\mathbb{C}}\mathrm{id}_{Y_1'}\big]\circ s$ and

$\widetilde{s} \circ f_2 = \pi_{X_2}$. Straightforward verification (based on the universality properties of the two pushout diagrams involved) shows that $\widetilde{r} \circ \widetilde{s} = \mathrm{id}_{Y_2}$. This suffices to conclude that $\widetilde{r}$ is surjective. Consequently the homomorphism f_2 has a separable type. $\qquad\square$

**Definition 6.4.** A characteristic $*$-homomorphism of a commutative square diagram $X_1 X_2 Y_1 Y_2$ is the $*$-homomorphism $\chi = f_2 \bigstar j$
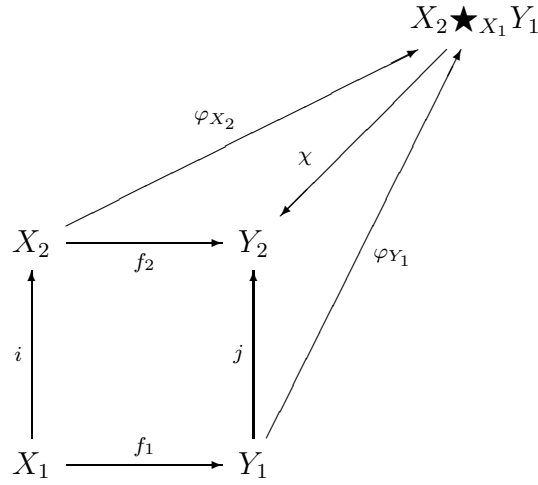
Note that a commutative square diagram is a pushout if and only if its characteristic $*$-homomorphism is an isomorphism.

Definition 6.5. A commutative square diagram, consisting of unital C^*-algebras and unital $*$-homomorphisms, is called doubly projective, if its characteristic $*$-homomorphism is doubly projective.

Lemma 6.6. *Let*

$$
\begin{array}{ccc}
X_2 & \xrightarrow{\ f_2\ } & Y_2 \\
{\scriptstyle i}\big\uparrow & & \big\uparrow{\scriptstyle j} \\
X_1 & \xrightarrow{\ f_1\ } & Y_1
\end{array}
$$

be a doubly projective square diagram. If f_1 is doubly projective, then f_2 is also doubly projective. Moreover, for any unital surjective $*$-homomorphism $p\colon A \to B$ of unital C^*-algebras and any three unital $*$-homomorphisms $g\colon X_2 \to A$, $h\colon Y_2 \to B$ and $k_1\colon Y_1 \to A$ such that $h \circ f_2 = p \circ g$, $g \circ i = k_1 \circ f_1$ and $h \circ j = p \circ k_1$,
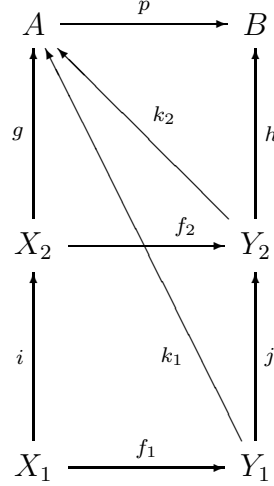
there exists a unital ∗-homomorphism $k_2 \colon Y_2 \to A$ such that $k_2 \circ f_2 = g$, $p \circ k_2 = h$ and $k_2 \circ j = k_1$.

Proof. Consider the pushout diagram $X_1 X_2 X_2 \bigstar_{X_1} Y_2 Y_2$ generated by the ∗-homomorphisms $i \colon X_1 \to X_2$ and $f_1 \colon X_1 \to Y_1$. Since f_1 is doubly projective, it follows, by Lemma 6.1, that φ_{X_2} is also doubly projective. Since the characteristic ∗-homomorphism $\chi \colon X_2 \bigstar_{X_1} X_1 \to Y_2$ of the originally given diagram is doubly projective, it follows, by Lemma 5.4, that the composition $f_2 = \chi \circ \varphi_{X_2}$ is doubly projective. This proves the first part of our statement.

In order to prove the second part of Lemma consider the following diagram in which all objects satisfy the above formulated assumptions:
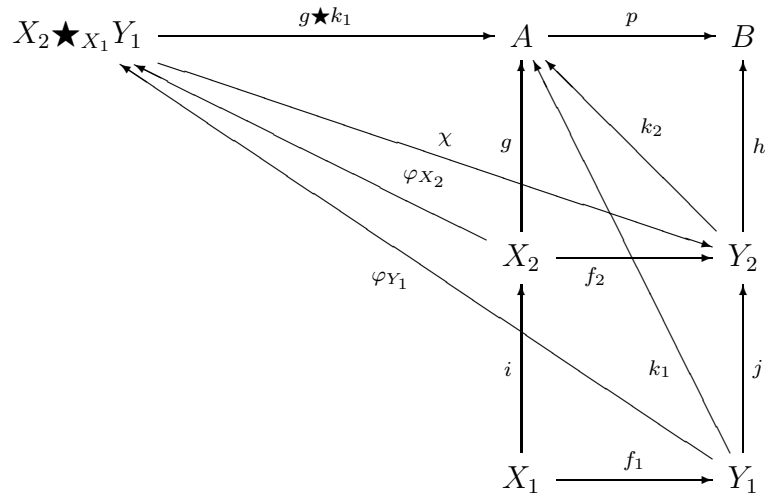
Since $X_1 X_2 X_2 \bigstar_{X_1} Y_2 Y_2$ is a pushout diagram and since the $*$-homomorphisms $g \colon X_2 \to A$ and $k_1 \colon Y_1 \to A$ satisfy the equality $g \circ i = k_1 \circ f_1$, there exists unique $*$-homomorphism $g \bigstar k_1 \colon X_2 \bigstar_{X_1} Y_1 \to A$ such that

$$(6.14) \qquad\qquad g = (g \bigstar k_1) \circ \varphi_{X_2}$$

and

$$(6.15) \qquad\qquad k_1 = (g \bigstar k_1) \circ \varphi_{Y_1}.$$

In order to prove that $p \circ (g \bigstar k_1) = h \circ \chi$, first observe that

$$(6.16) \qquad [p \circ (g \bigstar k_1)] \circ \varphi_{Y_1} = p \circ k_1 = h \circ j = [h \circ \chi] \circ \varphi_{Y_1}.$$

Secondly,

$$(6.17) \qquad [p \circ (g \bigstar k_1)] \circ \varphi_{X_2} = p \circ g = h \circ f_2 = [h \circ \chi] \circ \varphi_{X_2}.$$

Since $X_1 X_2 X_2 \bigstar_{X_1} Y_2 Y_2$ is a pushout diagram, (6.16) and (6.17), imply the required equality $p \circ (g \bigstar k_1) = h \circ \chi$.

Since χ is doubly projective the latter equality guarantees the existence of a unital $*$-homomorphism $k_2 \colon Y_2 \to A$ such that $p \circ k_2 = h$ and $k_2 \circ \chi = g \bigstar k_1$. The straitforward verification shows that $k_2 \circ f_2 = g$ and $k_2 \circ j = k_1$

$$k_2 \circ f_2 = k_2 \circ \chi \circ \varphi_{X_2} = (g \bigstar k_1) \circ \varphi_{X_2} \overset{(6.14)}{=} g$$

and

$$k_2 \circ j = k_2 \circ \chi \circ \varphi_{Y_1} = (g \bigstar k_1) \circ \varphi_{Y_1} \overset{(6.15)}{=} k_1.$$

This completes the proof of Lemma 6.6. $\square$

Proposition 6.7. *Let $\mathcal{S}_X = \{X_\alpha, i_\alpha^{\alpha+1}, \tau\}$ and $\mathcal{S}_Y = \{Y_\alpha, j_\alpha^{\alpha+1}, \tau\}$ be two well ordered continuous direct systems consisting of unital C^*-algebras and unital $*$-homomorphisms. Let*

$$\{f_\alpha \colon X_\alpha \to Y_\alpha; \alpha \in \tau\} \colon \mathcal{S}_X \to \mathcal{S}_Y$$

be a morphism between these systems such that all arising adjacent square diagrams

$$
\begin{array}{ccc}
X_{\alpha+1} & \xrightarrow{\;f_{\alpha+1}\;} & Y_{\alpha+1} \\[4pt]
{\scriptstyle i_\alpha^{\alpha+1}}\big\uparrow & & \big\uparrow{\scriptstyle j_\alpha^{\alpha+1}} \\[4pt]
X_\alpha & \xrightarrow{\;f_\alpha\;} & Y_\alpha
\end{array}
$$

are doubly projective. If $f_0 \colon X_0 \to Y_0$ is doubly projective, then the limit homomorphism $\varinjlim\{f_\alpha\} \colon \varinjlim \mathcal{S}_X \to \mathcal{S}_Y$ is also doubly projective.

Proof. Let $p\colon A \to B$ be a unital surjective $*$-homomorphism of unital C^*-algebras. Consider two unital $*$-homomorphisms

$$g\colon \varinjlim \mathcal{S}_X \to A \text{ and } h\colon \varinjlim \mathcal{S}_Y \to B$$

such that $p \circ g = h \circ \varinjlim\{f_\alpha\}$. Our goal is to construct a unital $*$-homomorphism

$$k\colon \varinjlim \mathcal{S}_Y \to A$$

such that $k \circ \varinjlim\{f_\alpha\} = g$ and $p \circ k = h$. Let

$$g_\alpha = g \circ i_\alpha\colon X_\alpha \to A \text{ and } h_\alpha = h \circ j_\alpha\colon Y_\alpha \to B, \alpha < \tau.$$

We now construct (by induction) a collection of unital $*$-homomorphisms

$$k_\alpha\colon Y_\alpha \to A, \ \alpha < \tau,$$

so that the following conditions are satisfied:

(a) $g_\alpha = k_\alpha \circ f_\alpha$, $\alpha < \tau$.
(b) $h_\alpha = p \circ k_\alpha$, $\alpha < \tau$.
(c) $k_\alpha = k_{\alpha+1} \circ j_\alpha^{\alpha+1}$, $\alpha < \tau$.
(d) $k_\alpha = \varinjlim\{k_\beta; \beta < \alpha\}$, whenever α is a limit ordinal number with $\alpha < \tau$.

By our assumption, the $*$-homomorphism f_0 is doubly projective. Consequently there exists a unital $*$-homomorphism $k_0\colon Y_0 \to A$ such that $g_0 = k_0 \circ f_0$ and $h_0 = p \circ k_0$.

Suppose that for each $\alpha < \gamma$, where $\gamma < \tau$, we have already constructed unital $*$-homomorphisms $k_\alpha\colon Y_\alpha \to A$ satisfying conditions (a)–(d) for appropriate indices. Let us construct a unital $*$-homomorphism $k_\gamma\colon Y_\gamma \to A$.

If γ is a limit ordinal number, then let (consult with Subsection 3.1)

$$k_\gamma = \varinjlim\{f_\alpha\colon \alpha < \gamma\}.$$

The continuity of the direct systems $\mathcal{S}_X$ and $\mathcal{S}_Y$ guarantees that $g_\gamma = k_\gamma \circ f_\gamma$, $h_\gamma = p \circ k_\gamma$ and $k_\alpha = j_\alpha^\gamma \circ k_\gamma$ for each $\alpha < \gamma$.

If $\gamma = \alpha + 1$, then, by the assumption, the diagram

$$
\begin{array}{ccc}
X_{\alpha+1} & \xrightarrow{\ f_{\alpha+1}\ } & Y_{\alpha+1} \\[4pt]
{\scriptstyle i_\alpha^{\alpha+1}}\big\uparrow & & \big\uparrow{\scriptstyle j_\alpha^{\alpha+1}} \\[4pt]
X_\alpha & \xrightarrow{\ f_\alpha\ } & Y_\alpha
\end{array}
$$

is doubly projective. Therefore, by Lemma 6.6, there exists a unital $*$-homomorphism $k_{\alpha+1}\colon Y_{\alpha+1} \to A$ such that $g_{\alpha+1} = k_{\alpha+1} \circ f_{\alpha+1}$, $h_{\alpha+1} = p \circ k_{\alpha+1}$ and $k_\alpha = j_\alpha^{\alpha+1} \circ k_{\alpha+1}$.

Thus, the $*$-homomorphisms k_α are now constructed for each $\alpha < \tau$. It only remains to note that the $*$-homomorphism $k = \varinjlim\{k_\alpha\}\colon \varinjlim \mathcal{S}_Y \to A$ satisfies all the required properties. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6.2. **Characterization of doubly projective homomorphisms.**

Theorem 6.8. *Let $f\colon X \to Y$ be a unital $*$-homomorphism between unital C^*-algebras of the same density. Then f is doubly projective homomorphism of separable type if and only if there exist direct C^*_ω-systems $\mathcal{S}_X = \{X_\alpha, i_\alpha, \mathcal{A}\}$, $\mathcal{S}_Y = \{Y_\alpha, j^\beta_\alpha, \mathcal{A}\}$ and a morphism $\{f_\alpha\colon X_\alpha \to Y_\alpha; \alpha \in \mathcal{A}\}\colon \mathcal{S}_X \to \mathcal{S}_Y$, satisfying the following conditions:*

(a) *The indexing set $\mathcal{A}$ is cofinal and ω-closed in $\exp_\omega \tau$.*
(b) *$X = \varinjlim \mathcal{S}_X$, $Y = \varinjlim \mathcal{S}_Y$, $f = \varinjlim\{f_\alpha; \alpha \in \mathcal{A}\}$.*
(c) *X_α and Y_α are separable unital projective C^*-algebras, $\alpha \in \mathcal{A}$.*
(d) *The α-th limit inclusions $i_\alpha\colon X_\alpha \to X$ and $j_\alpha\colon Y_\alpha \to Y$ are doubly projective, $\alpha \in \mathcal{A}$.*
(e) *$f_\alpha\colon X_\alpha \to Y_\alpha$ is doubly projective, $\alpha \in \mathcal{A}$.*
(f) *All α-th limit diagrams ($\alpha \in \mathcal{A}$)*

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\[2pt]
{\scriptstyle i_\alpha}\big\uparrow & & \big\uparrow{\scriptstyle j_\alpha} \\[2pt]
X_\alpha & \xrightarrow{\ f_0\ } & Y_\alpha,
\end{array}
$$

are pushouts.

Proof. Part I. Let $f\colon X \to Y$ be a doubly projective homomorphism of separable type. We will show the existence of the above indicated direct C^*_ω-systems and of a morphism, satisfying the required properties.

If Y is separable, then the statement is trivial. Indeed, by Lemma 5.2, f is injective and consequently X is also separable. Let $X_0 = X$, $Y_0 = Y$, $p = \mathrm{id}_X$, $q = \mathrm{id}_Y$ and $f_0 = f$. Obviously the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\[2pt]
{\scriptstyle \mathrm{id}_X}\big\uparrow & & \big\uparrow{\scriptstyle \mathrm{id}_Y} \\[2pt]
X & \xrightarrow{\ f\ } & Y,
\end{array}
$$

is a pushout.

Now consider the case $d(Y) = \tau > \omega$. By our assumption, the homomorphism f has a separable type. This means (see Definition 5.8) that there exist a projective unital C^*-algebra Z such that $d(Z) = d(X)$, a separable projective unital C^*-algebra K and two surjective unital $*$-homomorphisms $\varphi_X\colon Z \to X$

and $\varphi_Y \colon Z \bigstar_{\mathbb{C}} K \to Y$ such that $f \circ \varphi_X = \varphi_Y \circ \pi_Z$, where $\pi_Z \colon Z \to Z \bigstar_{\mathbb{C}} K$ denotes the natural inclusion. In other words, the following diagram

$$
\begin{array}{ccc}
Y & \xleftarrow{\varphi_Y} & Z \bigstar_{\mathbb{C}} K \\
{\scriptstyle f}\big\uparrow & & \big\uparrow{\scriptstyle \pi_Z} \\
X & \xleftarrow{\varphi_X} & Z
\end{array}
$$

commutes.

Since X is projective and $\varphi_X \colon Z \to X$ is surjective, there exists a unital $*$-homomorphism $\phi_X \colon X \to Z$ such that $\varphi_X \circ \phi_X = \mathrm{id}_X$. Now consider the square diagram

$$
\begin{array}{ccc}
Y & \xleftarrow{\mathrm{id}_Y} & Y \\
{\scriptstyle \varphi_Y}\big\uparrow \quad {\scriptstyle \phi_Y}\nearrow & & \big\uparrow{\scriptstyle f} \\
Z \bigstar_{\mathbb{C}} K & \xleftarrow{\pi_Z \circ \phi_X} & X
\end{array}
$$

which obviously commutes. To see this note that

$$
\varphi_Y \circ \pi_Z \circ \phi_X = f \circ \varphi_X \circ \phi_X = f.
$$

Since φ_Y is surjective and since f is doubly projective, there exists a unital $*$-homomorphism $\phi_Y \colon Z \bigstar_{\mathbb{C}} K \to Y$ (indicated in the above diagram as the diagonal arrow) such that $\varphi_Y \circ \phi_Y = \mathrm{id}_Y$ and $\phi_Y \circ f = \pi_Z \circ \phi_X$. Thus we have the commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\phi_Y} & Z \bigstar_{\mathbb{C}} K \\
{\scriptstyle f}\big\uparrow & & \big\uparrow{\scriptstyle \pi_Z} \\
X & \xrightarrow{\phi_X} & Z.
\end{array}
$$

Next observe that the C^*-algebras X, Y, Z and $Z \bigstar_{\mathbb{C}} K$ all have density $\leq \tau$. Consequently, by Theorem 5.13, $X = \varinjlim \mathcal{S}_X$, $Y = \varinjlim \mathcal{S}_Y$ and $Z = \varinjlim \mathcal{S}_Z$, where $\mathcal{S}_X = \{X_\alpha, i_\alpha^\beta, \mathcal{A}_X\}$, $\mathcal{S}_Y = \{Y_\alpha, j_\alpha^\beta, \mathcal{A}_Y\}$ and $\mathcal{S}_Z = \{Z_\alpha, s_\alpha^\beta, \mathcal{A}_Z\}$ are direct C^*_ω-systems consisting of separable unital projective C^*-algebras and doubly projective limit inclusions $i_\alpha \colon X_\alpha \to X$, $\alpha \in \mathcal{A}_X$, $j_\alpha \colon Y_\alpha \to Y$, $\alpha \in \mathcal{A}_Y$, and $s_\alpha \colon Z_\alpha \to Z$, $\alpha \in \mathcal{A}_Z$. Also note that all three indexing sets $\mathcal{A}_X$, $\mathcal{A}_Y$ and $\mathcal{A}_Z$ are cofinal and ω-closed subsets of $\exp_\omega \tau$. Next observe that the unital free product $Z \bigstar_{\mathbb{C}} K$ is also the limit of the direct system $\mathcal{S}_{Z \bigstar_{\mathbb{C}} K} = \{Z_\alpha \bigstar_{\mathbb{C}} K, s_\alpha^\beta \bigstar_{\mathbb{C}} \mathrm{id}_K, \mathcal{A}_Z\}$

(straightforward verification using the universality properties of unital free products and limits of direct systems; see also Section 4). An important consequence of the fact that f has a separable type is that K is a separable C^*-algebra. This guarantees, according to [1, Theorem 3.1], that each C^*-algebra $Z_\alpha \bigstar_\mathbb{C} K$, $\alpha \in \mathcal{A}_Z$, is separable and, as a result, $\mathcal{S}_{Z\bigstar_\mathbb{C}K}$ is actually a direct C^*_ω-system.

For each $\alpha \in \mathcal{A}_Z$ let $\widetilde{X}_\alpha = \mathrm{cl}_X\left(\varphi_X(Z_\alpha)\right)$. Let also $\widetilde{i}^\beta_\alpha \colon \widetilde{X}_\alpha \to \widetilde{X}_\beta$, $\alpha \leq \beta$, $\alpha, \beta \in \mathcal{A}_Z$ denote the corresponding inclusion. Similarly, for each $\alpha \in \mathcal{A}_Z$ let $\widetilde{Y}_\alpha = \mathrm{cl}_Y\left(\varphi_Y(Z_\alpha \bigstar_\mathbb{C}K)\right)$ and $\widetilde{j}^\beta_\alpha \colon \widetilde{Y}_\alpha \to \widetilde{Y}_\beta$, $\alpha \leq \beta$, $\alpha, \beta \in \mathcal{A}_Z$ denote the corresponding inclusion. It is easy to see that the systems $\widetilde{\mathcal{S}}_X = \{\widetilde{X}_\alpha, \widetilde{i}^\beta_\alpha, \mathcal{A}_Z\}$ and $\widetilde{\mathcal{S}}_Y = \{\widetilde{Y}_\alpha, \widetilde{j}^\beta_\alpha, \mathcal{A}_Z\}$ are direct C^*_ω-systems such that $\varinjlim \widetilde{\mathcal{S}}_X = X$ and $\varinjlim \widetilde{\mathcal{S}}_Y = Y$.

Since the indexing sets $\mathcal{A}_X$, $\mathcal{A}_Y$ and $\mathcal{A}_Z$ are cofinal and ω-closed in $\exp_\omega \tau$, we can conclude, by Proposition 2.1, that the intersection $\mathcal{B} = \mathcal{A}_X \cap \mathcal{A}_Y \cap \mathcal{A}_Z$ is still cofinal and ω-closed in $\exp_\omega \tau$.

Next we consider six homomorphisms

$$\varphi_X \colon \varinjlim \mathcal{S}_Z|\mathcal{B} \to \varinjlim \mathcal{S}_X|\mathcal{B}, \qquad\qquad \varphi_Y \colon \varinjlim \mathcal{S}_{Z\bigstar_\mathbb{C}K}|\mathcal{B} \to \varinjlim \mathcal{S}_Y|\mathcal{B},$$

$$\phi_X \colon \varinjlim \mathcal{S}_X|\mathcal{B} \to \varinjlim \mathcal{S}_Z|\mathcal{B}, \qquad\qquad \phi_Y \colon \varinjlim \mathcal{S}_Y|\mathcal{B} \to \varinjlim \mathcal{S}_{Z\bigstar_\mathbb{C}K}|\mathcal{B},$$

$$\pi_Z \colon \varinjlim \mathcal{S}_Z|\mathcal{B} \to \varinjlim \mathcal{S}_{Z\bigstar_\mathbb{C}K}|\mathcal{B} \qquad \text{and} \qquad f \colon \varinjlim \mathcal{S}_X|\mathcal{B} \to \varinjlim \mathcal{S}_Y|\mathcal{B}.$$

Three of these homomorphisms are, by construction, the limits of associated morphisms

$$\varphi_X = \varinjlim\{\varphi_X^\alpha \colon Z_\alpha \to X_\alpha; \mathcal{B}\}, \quad \text{where} \quad \varphi_X^\alpha = \varphi_X|Z_\alpha, \quad \alpha \in \mathcal{B},$$

$$\varphi_Y = \varinjlim\{\varphi_Y^\alpha \colon Z_\alpha \bigstar_\mathbb{C}K \to Y_\alpha; \mathcal{B}\}, \quad \text{where} \quad \varphi_Y^\alpha = \varphi_Y|\left(Z_\alpha \bigstar_\mathbb{C}K\right), \quad \alpha \in \mathcal{B},$$

and

$$\pi_Z = \varinjlim\{\pi_{Z_\alpha} \colon Z_\alpha \to Z_\alpha \bigstar_\mathbb{C}K; \mathcal{B}\}, \quad \text{where } \pi_{Z_\alpha} \text{ is the canonical inclusion}, \ \alpha \in \mathcal{B},$$

We apply Theorem 3.5 to the remaining three homomorphisms φ_X, φ_Y and f and conclude that there exist cofinal and ω-complete subsets $\mathcal{B}_{\varphi_X}$, $\mathcal{B}_{\varphi_Y}$ and $\mathcal{B}_f$ of $\mathcal{B}$ and morphisms

$$\{\varphi_X^\alpha \colon X_\alpha \to Z_\alpha; \mathcal{B}_{\varphi_X}\} \colon \mathcal{S}_X|\mathcal{B}_{\varphi_X} \to \mathcal{S}_Z|\mathcal{B}_{\varphi_X},$$

$$\{\varphi_Y^\alpha \colon Y_\alpha \to Z_\alpha \bigstar_\mathbb{C}K; \mathcal{B}_{\varphi_Y}\} \colon \mathcal{S}_Y|\mathcal{B}_{\varphi_X} \to \mathcal{S}_{Z\bigstar_\mathbb{C}K}|\mathcal{B}_{\varphi_Y},$$

and

$$\{f_\alpha \colon X_\alpha \to Y_\alpha; \mathcal{B}_f\} \colon \mathcal{S}_X|\mathcal{B}_f \to \mathcal{S}_Y|\mathcal{B}_f$$

such that

$$\varphi_X = \varinjlim\{\varphi_X^\alpha; \alpha \in \mathcal{B}_{\varphi_X}\}, \ \varphi_Y = \varinjlim\{\varphi_Y^\alpha; \alpha \in \mathcal{B}_{\varphi_Y}\} \text{ and } f = \varinjlim\{f_\alpha; \alpha \in \mathcal{B}_f\}.$$

Note that, by Proposition 2.1, the intersection $\mathcal{A} = \mathcal{B}_{\varphi_X} \cap \mathcal{B}_{\varphi_Y} \cap \mathcal{B}_f$ is cofinal and ω-closed in $\mathcal{B}$ (and consequently in $\exp_\omega \tau$).

For each $\alpha \in \mathcal{A}$ we have the following commutative diagram:

Note that, by Theorem 5.13, we may without loss of generality assume that the limit inclusions $i_\alpha \colon X_\alpha \to X$ and $j_\alpha \colon Y_\alpha \to Y$, $\alpha \in \mathcal{A}$, are doubly projective. This observation coupled with Lemma 5.5 guarantees that the homomorphism $f_\alpha \colon X_\alpha \to Y_\alpha$, $\alpha \in \mathcal{A}$, is also doubly projective.

It is now clear that in order to complete the proof it suffices to show that the diagram (the front face of the above cubic diagram)

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
i_\alpha \uparrow & & \uparrow j_\alpha \\
X_\alpha & \xrightarrow{\ f\alpha\ } & Y_\alpha
\end{array}
$$

is a pushout for an arbitrary index $\alpha \in \mathcal{A}$. Let $p \colon X \to R$ and $q \colon Y_\alpha \to R$ be unital $*$-homomorphisms into a unital C^*-algebra R such that $p \circ i_\alpha = q \circ f_\alpha$. Consider the homomorphisms $\widetilde{p} = p \circ \varphi_X \colon Z \to R$ and $\widetilde{q} = q \circ \varphi_Y^\alpha \colon Z_\alpha \bigstar_\mathbb{C} K \to R$. Note that

$$\widetilde{p} \circ s_\alpha = p \circ \varphi_X \circ s_\alpha = p \circ i_\alpha \circ \varphi_X^\alpha = q \circ f_\alpha \circ \varphi_X^\alpha = q \circ \varphi_Y^\alpha \circ \pi_{Z_\alpha} = \widetilde{q} \circ \pi_{Z_\alpha}.$$

Now let $\alpha \in \mathcal{A}$. Since, by Lemma 6.2, the diagram (the back face of the above cubic diagram)

$$Z \xrightarrow{\pi_Z} Z \bigstar_{\mathbb{C}} K$$

$$s_\alpha \uparrow \qquad\qquad \uparrow s_\alpha \bigstar_{\mathbb{C}} \mathrm{id}_K$$

$$Z_\alpha \xrightarrow{\pi_{Z_\alpha}} Z_\alpha \bigstar_{\mathbb{C}} K$$

is a pushout, it follows that there exists a unique unital $*$-homomorphism $\widetilde{r} \colon Z \bigstar_{\mathbb{C}} K \to R$ such that $\widetilde{p} = \widetilde{r} \circ \pi_Z$ and $\widetilde{q} = \widetilde{r} \circ (s_\alpha \bigstar_{\mathbb{C}} \mathrm{id}_K)$. Now let $r = \widetilde{r} \circ \phi_Y \colon Y \to R$. We have

$$r \circ j_\alpha = \widetilde{r} \circ \phi_Y \circ j_\alpha = \widetilde{r} \circ (s_\alpha \bigstar_{\mathbb{C}} \mathrm{id}_K) \circ \phi_Y^\alpha = \widetilde{q} \circ \phi_Y^\alpha = q \circ \varphi_Y^\alpha \circ \phi_Y^\alpha = q$$

and

$$r \circ f = \widetilde{r} \circ \phi_Y \circ f = \widetilde{r} \circ \pi_Z \circ \phi_X = \widetilde{p} \circ \phi_X = p \circ \varphi_X \circ \phi_X = p.$$

This simply means that the diagram under consideration has the corresponding universality property. Finally the uniqueness of $\widetilde{r}$ guarantees that r is the only unital $*$-homomorphism with the just indicated properties. This shows that our diagram is pushout and completes the proof of part I.

Part II. Suppose that we are given direct C_ω^*-systems $\mathcal{S}_X = \{X_\alpha, i_\alpha^\beta, \mathcal{A}\}$, $\mathcal{S}_Y = \{Y_\alpha, j_\alpha^\beta, \mathcal{A}\}$ and a morphism $\{f_\alpha \colon X_\alpha \to Y_\alpha; \alpha \in \mathcal{A}\} \colon \mathcal{S}_X \to \mathcal{S}_Y$, satisfying the above indicated properties.

Let $\alpha \in \mathcal{A}$. By conditions (d), (e) and Lemma 5.4, the composition $j_\alpha \circ f_\alpha$ is doubly projective. Since, by condition (d), the inclusion i_α is doubly projective, it follows from Lemmas 5.2 and 5.5, that f is also doubly projective. By condition (c) and Lemma 5.9, the homomorphism f_α has a separable type. Finally, by condition (f) and Lemma 6.3, f also has a separable type. $\qquad\square$

Corollary 6.9. *Let $f \colon X \to Y$ be a doubly projective homomorphism of unital projective C^*-algebras. If f has a separable type, then there exists a pushout*

$$X \xrightarrow{f} Y$$

$$p \uparrow \qquad\qquad \uparrow q$$

$$X_0 \xrightarrow{f_0} Y_0,$$

where X_0 and Y_0 are separable unital projective C^*-algebras and the homomorphisms $i_0 \colon X_0 \to Y_0$, $p \colon X_0 \to X$ and $q \colon Y_0 \to Y$ are doubly projective.

Remark 6.10. Combining methods of proofs of Theorems 5.13 and 6.8 it is possible to obtain a characterization of arbitrary (not necessarily of a separable type) doubly projective homomorphisms of unital C^*-algebras. This characterization is recorded in Theorem 6.11. We only note here that the sufficiency follows from Proposition 6.7.

Theorem 6.11. *A unital $*$-homomorphism $f\colon X \to Y$ of projective unital C^*-algebras is doubly projective if and only if there exist well ordered continuous direct systems $\mathcal{S}_X = \{X_\alpha, i_\alpha^{\alpha+1}, \tau\}$, $\mathcal{S}_Y = \{Y_\alpha, j_\alpha^{\alpha+1}, \tau\}$ and a morphism $\{f_\alpha; \tau\}\colon \mathcal{S}_X \to \mathcal{S}_Y$ satisfying the following conditions:*

(a) *$X = \varinjlim \mathcal{S}_X$, $Y = \varinjlim \mathcal{S}_Y$ and $f = \varinjlim \{f_\alpha; \tau\}$.*

(b) *C^*-algebras X_0 and Y_0 are separable projective and the homomorphism $f_0\colon X_0 \to Y_0$ is doubly projective.*

(c) *C^*-algebras X_α and Y_α are projective and the homomorphism $f_\alpha\colon X_\alpha \to Y_\alpha$ is doubly projective, $\alpha < \tau$.*

(d) *All short injections $i_\alpha^{\alpha+1}\colon X_\alpha \to X_{\alpha+1}$ and $j_\alpha^{\alpha+1}\colon Y_\alpha \to Y_{\alpha+1}$ are doubly projective and have a separable type.*

(e) *All adjacent square diagrams*

$$
\begin{array}{ccc}
X_{\alpha+1} & \xrightarrow{\ f_{\alpha+1}\ } & Y_{\alpha+1} \\[2pt]
{\scriptstyle i_\alpha^{\alpha+1}}\Big\uparrow & & \Big\uparrow{\scriptstyle j_\alpha^{\alpha+1}} \\[2pt]
X_\alpha & \xrightarrow{\ f_\alpha\ } & Y_\alpha
\end{array}
$$

are doubly projective and their characteristic homomorphisms have separable type.

(f) *If the homomorphism f itself has a separable type, then all the square diagrams indicated in (d) are pushouts.*

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